# A Pseudoinverse-Based Iterative Learning Control

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***Abstract*—Learning control is a very effective approach for tracking control in processes occuring repetitively over a fixed interval of time. In this note, an iterative learning control (ILC) algorithm is proposed to accommodate a general class of nonlinear, nonminimum-phase plants with disturbances and initialization errors. The algorithm requires the computation of an approximate inverse of the linearized plant rather than the exact inverse. An advantage of this approach is that the output of the plant need not be differentiated. A bound on the asymptotic trajectory error is exhibited via a concise proof and is shown to grow continuously with a bound on the disturbances. The structure of the controller is such that the low frequency components of the trajectory converge faster than the high frequency components.**

***Index Terms*—Iterative learning control (ILC), nonlinear tracking, pseudo inverse.**

## I. Introduction

Iterative learning control (ILC) refers to a class of self-tuning controllers where the system performance of a specified task is gradually improved or perfected based on the previous performances of identical tasks. The most common applications of learning control are in the area of robot control in production industries, where a robot is required to perform a single task, say pick-and-place an object along a given trajectory, repetitively. With a feedback controller alone, the same tracking error would persist in every repeated trial. In contrast, a learning controller can use the information from the previous execution to improve the tracking performance in the next execution. While in some applications, the need to repeat a trajectory multiple times for learning may be a disadvantage, we focus our attention on those many others where learning control is a natural solution.

In this note, we propose a modification of the iterative learning control algorithm presented in [1] so that it can be applied to a more generic class of nonlinear nonminimum phase plants with input disturbance and output sensor noise. In Section II, a learning controller is proposed by formulating a pseudo-inverse of the linearized plant at the origin. In Section III, simulation examples are presented to show the performance of the proposed learning controller. Finally, Section IV concludes the note.

## II. Nonlinear Nonminimum Phase Plant With Disturbances

In this section, we present a robust iterative learning algorithm for nonlinear systems. We consider only square (same number of inputs and outputs) time-invariant nonlinear systems.

### A. System Description

Consider a nonlinear system which is *stable-in-first-approximation* at $x = 0$ (i.e., the linearized plant has all the eigenvalues in the open left half complex plane) and also input-to-state stable

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + g(x_i(t))\, u_i(t) + b(x_i(t))\, w_i(t), \;\; x_i(0) = 0 \\ y_i(t) &= h(x_i(t)) + v_i(t) \end{aligned} \tag{1}$$

where $i$ is the index of iteration of ILC, $\{u_i\}_{i=0}^{\infty}$ is a family of input sequence, $x_i(t) \in \mathbb{R}^n$, $u_i(t)$ and $y_i(t) \in \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^m$, $b : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, $w_i \in \mathbb{R}^p$, $v_i \in \mathbb{R}^m$. The function $w_i(t)$ represents both repetitive and random bounded disturbances of the system; it may be stiction, nonreproducible friction, state-independent modeling errors, etc., $v_i(t)$ represents sensor noise. A desired trajectory $y_d(t)$ is supported on finite interval ($t \in [0, T]$) of the time axis. The objective of learning is to construct a sequence of input trajectories $\{u_i\}_{i=1}^{\infty}$ such that $u_i \to u^*$ and $u^*(t)$ causes the system to track a trajectory $y_d(t)$ "as closely as possible" on $[0, T]$ We make the following assumptions:

($\mathcal{A}1$) The functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are continuously differentiable and $b(\cdot)$ is continuous.

($\mathcal{A}2$) $u_0 \in L_\infty \cap C^0 \cap B_r$, where $B_r$ is a closed subset of a Banach space.

($\mathcal{A}3$) The system is *stable-in-first-approximation* and input-to-state stable.

(Note: If the system is not stable, it may be stabilized prior to application of our methods).

($\mathcal{A}4$) The disturbances $w_i(\cdot)$ and $v_i(\cdot)$ are bounded by $b_w$ and $b_v$, respectively (i.e., $\|w_i(t)\| \le b_w$ and $\|v_i(t)\| \le b_v$).

($\mathcal{A}5$) The desired trajectory $y_d$ lies sufficiently close to a trajectory $\hat{y}_d$ which satisfies the following equations:

$$\begin{aligned} \dot{\hat{x}}_d(t) &= f(\hat{x}_d(t)) + g(\hat{x}_d(t))\hat{u}_d(t) \quad \hat{x}_d(0) = 0 \\ \hat{y}_d(t) &= h(\hat{x}_d(t)), \quad \forall t \in [0, T]. \end{aligned} \tag{2}$$

For such a system, an ILC is proposed as shown in Fig. 1.

### B. Formulation of Learning Controller

In this section, a good candidate for the learning controller $LC$ of Fig. 1 is derived by first linearizing the plant and then a "pseudoinverse" of the linearized plant is used as the learning controller.

The update law of the ILC is written in terms of the operators $P$, the linearized plant $DP|_0$, its adjoint $DP|_0^*$ and $\mathcal{T}$ for $t \in [0, T]$ is

$$\begin{aligned} u_{i+1}(t) &= \mathcal{T}(u_i + \delta u_i) \\ &= \mathcal{T}\left(u_i + \underbrace{(\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*}_{\text{pseudoinverse}, \; DP|_0^{\dagger,\alpha}} (y_d - P(u_i))\right) \\ &= \mathcal{T}\left(u_i + (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^* (y_d - y_i)\right). \end{aligned} \tag{3}$$

Note that if $u_0 = \mathcal{T}(u_0)$, $\mathcal{T}(u_i(t) + \delta u_i(t)) = u_i(t) + \mathcal{T}(\delta u_i(t))$ for all $i$. (Note that in Fig. 1, the truncation operator $\mathcal{T}$ is placed before the summing junction).

*Defining $DP|_0$:* Since the nonlinear system (1) is input-to-state-stable ($\mathcal{A}5$) and $h$ is continuous ($\mathcal{A}1$), it defines a causal nonlinear input-to-output map $P$ as: $P : u_i \mapsto y_i;\; L_\infty[0, \infty) \to L_\infty[0, \infty)$. Since $P$ is *stable-in-first-approximation* ($\mathcal{A}5$), we define a stable time-invariant input-to-output linear operator $DP|_0$ by linearizing the system (1) around ($x_i = 0$, $u_i = 0$, $w_i = 0$, $v_i = 0$) as follows:

$$\begin{aligned} \dot{\delta x}(t) &= A\delta x(t) + B\delta u(t) \quad \delta x(0) = 0 \\ \delta y(t) &= C\delta x(t). \end{aligned} \tag{4}$$

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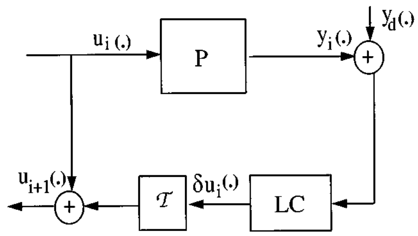



Fig. 1. Nonlinear learning control system $P$: nonlinear plant, $LC$. Learning controller $\mathcal{T}$: truncation operator.

where $A \triangleq f_x(0)$, $B \triangleq g(0)$, $C \triangleq h_x(0)$. Hence, $DP|_0 : \delta u \mapsto \delta y;\ L_\infty[0,\infty) \rightarrow L_\infty[0,\infty)$. Since $\delta u \in L_\infty[0,T]$ and $A$ is Hurwitz [in (4)] we can replace $\delta x(0) = 0$ with $\delta x(\pm\infty) = 0$ and not alter the input–output (I–O) map defined by (4), and hence, the only map provided is $1 - 1$.

*Defining $DP|_0^*$:* Consider the I–O map for the adjoint system

$$\begin{aligned}\delta\dot{\bar{x}} =& -A^T\delta\bar{x} - C^T\delta\bar{u}, \quad \delta\bar{x}(\pm\infty) = 0\\ \delta\bar{y} =& B^T\delta\bar{x}.\end{aligned} \tag{5}$$

Since $A$ is Hurwitz, $-A^T$ is hyperbolic (*i.e.*, none of the eigenvalues have zero real part). Further, (5) defines a unique noncausal mapping as shown by Devasia *et al.* [2] (see the Appendix ). $DP|_0^*$: $\delta\bar{u} \mapsto \delta\bar{y};\ L_\infty \rightarrow L_\infty$. The adjoint system satisfies the property $\langle DP|_0 u, v\rangle = \langle u, DP|_0^* v\rangle$ [3].

*Defining $DP|_{u_i}$:* Neglecting higher order terms we obtain a linearized plant around the solution $x_i(t)$ to (1)

$$\begin{aligned}\delta\dot{\tilde{x}}_i(t) =& f_x(x_i)\,\delta\tilde{x}_i(t) + g_x(x_i)\,\delta\tilde{x}_i(t)u_i(t)\\ &+ g(x_i)\,\delta u_i(t) + b(x_i)\,w_i(t)\\ \delta\tilde{x}_i(0) =& 0\\ \delta\tilde{y}_i(t) =& h_x(x_i)\,\delta\tilde{x}_i(t) + v_i(t)\end{aligned} \tag{6}$$

where $f_x(x_i(t)) \triangleq \partial f/\partial x(x_i(t))$; $g_x(x_i(t)) \triangleq \partial g/\partial x(x_i(t))$. Since (4) is stable, it can be proved by Lyapunov methods that (6) is also bounded-input–bounded-output (BIBO) stable if $x_i$ lies within a certain bound. Note that, here also we can replace $\delta\tilde{x}_i(0) = 0$ (as in (4)) with $\delta\tilde{x}_i(\pm\infty) = 0$ and not alter the I–O map. Define $A_i(t) \triangleq f_x(x_i(t)) + g_x(x_i(t))u_i(t)$, $B_i(t) \triangleq g(x_i(t))$, $C_i(t) \triangleq h_x(x_i(t))$, $b_i(t) \triangleq b(x_i(t))$. The stable linear system (6) has a solution and defines a linear I–O map: $DP|_{u_i} : \delta u_i \mapsto \delta y_i;\ L_\infty \rightarrow L_\infty$.

*Defining $DP|_0^{\dagger,\alpha}$:* Motivated by the concept of a pseudo-inverse [4] we define learning controller by the following linear operator:

$$DP|_0^{\dagger,\alpha} \triangleq (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^* \tag{7}$$

for $\alpha \neq 0$. We call this "approximate inverse" the $\alpha$-pseudo inverse of $DP|_0$. For simplicity the $\alpha$ pseudoinverse is referred to as simply a pseudo-inverse in the rest of this note. In time-domain using (4) and (5) $(\alpha I + DP|_0^* DP|_0)$: $\delta u \rightarrow \delta\tilde{y}$ is

$$\begin{aligned}\begin{bmatrix}\delta\dot{x}\\ \delta\dot{\bar{x}}\end{bmatrix} =& \begin{bmatrix}A & 0\\ -C^TC & -A^T\end{bmatrix}\begin{bmatrix}\delta x\\ \delta\bar{x}\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}\delta u\\ \begin{bmatrix}\delta x(\pm\infty)\\ \delta\bar{x}(\pm\infty)\end{bmatrix} =& 0\\ \delta\tilde{y} =& \alpha\delta u + \delta\bar{y} = \alpha\delta u + B^T\delta\bar{x}.\end{aligned} \tag{8}$$

Since $DP|_0$ is stable, (8) is hyperbolic with eigenvalues $\lambda(A) \cup \lambda(-A^T)$. Hence, by [2] $(\alpha I + DP|_0^* DP|_0) : \delta u \mapsto \delta\tilde{y};\ L_\infty \rightarrow L_\infty$ and is noncausal. Solving (8) for $\delta u$, we see that the inverted operator $(\alpha I + DP|_0^* DP|_0)^{-1}$ is

$$\begin{aligned}\begin{bmatrix}\delta\dot{x}\\ \delta\dot{\bar{x}}\end{bmatrix} =& \underbrace{\begin{bmatrix}A & -B\alpha^{-1}B^T\\ -C^TC & -A^T\end{bmatrix}}_{A_\alpha}\begin{bmatrix}\delta x\\ \delta\bar{x}\end{bmatrix}\\ &+ \begin{bmatrix}B\alpha^{-1}\\ 0\end{bmatrix}\delta\tilde{y}\ \begin{bmatrix}\delta x(\pm\infty)\\ \delta\bar{x}(\pm\infty)\end{bmatrix} = 0\\ \delta u =& \alpha^{-1}\left(\delta\tilde{y} - B^T\delta\bar{x}\right).\end{aligned} \tag{9}$$

The eigenvalues of the above system are continuous functions of $\alpha$. In the limit $\alpha \rightarrow \infty$, $A_\alpha$ is hyperbolic (since $A$ is Hurwitz). Thus, we can always choose an $\alpha$ for which $A_\alpha$ is hyperbolic. The system (9) is solved by the stable-noncausal-solution approach of Devasia *et al.* [2]. Hence, $(\alpha I + DP|_0^* DP|_0)^{-1} : \delta\tilde{y} \mapsto \delta u;\ L_\infty \rightarrow L_\infty$.

The **learning controller** is the pseudoinverse $(\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*$ and is given in time-domain by

$$\begin{aligned}\begin{bmatrix}\delta\dot{x}\\ \delta\dot{\bar{x}}\\ \delta\dot{z}\end{bmatrix} =& \underbrace{\begin{bmatrix}A & -B\alpha^{-1}B^T & B\alpha^{-1}B^T\\ -C^TC & -A^T & 0\\ 0 & 0 & -A^T\end{bmatrix}}_{A_c}\underbrace{\begin{bmatrix}\delta x\\ \delta\bar{x}\\ \delta z\end{bmatrix}}_{x}\\ &+ \begin{bmatrix}0\\ 0\\ -C^T\end{bmatrix}\delta y;\\ X(\pm\infty) =& 0\\ \delta u =& \begin{bmatrix}0 & -\alpha^{-1}B^T & \alpha^{-1}B^T\end{bmatrix}\begin{bmatrix}\delta x\\ \delta\bar{x}\\ \delta z\end{bmatrix}.\end{aligned} \tag{10}$$

$A_c$ is block diagonal, therefore the eigenvalues of $A_c$ are eigenvalues of $A_\alpha$ (9) and $-A^T$. Since $A_\alpha$ is hyperbolic for some $\alpha$, $A_c$ is hyperbolic. Hence, $(\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^* : \delta y \mapsto \delta u$, $L_\infty \rightarrow L_\infty$ and the solution of the linear controller described by (10) can be obtained using stable-noncausal-solution approach [2]. (Using initial conditions at $t = -\infty$ rather than $t = 0$ allows us to control $X(0)$ via $\delta y(-\infty, 0]$. Thus tracking performance can be improved relative to assumptions of $\delta y \equiv 0$ on $(-\infty, 0]$ and $X(0) = 0$).

### C. Convergence Analysis

*Definition 1:* We define the $\lambda$ norm for a function $x : [0,T] \rightarrow \mathbb{R}^k$ by

$$\|x(\cdot)\|_\lambda \triangleq \sup_{t\in[0,T]} e^{-\lambda t}\|x(t)\|. \tag{11}$$

Note that $\|x\|_\lambda \leq \|x\|_\infty \leq e^{\lambda T}\|x\|_\lambda$ for $\lambda > 0$, implying that $\|x\|_\lambda$ and $\|x\|_\infty$ are equivalent norms. Thus convergence results can be proved using either norm.

*Induced $\lambda$-norm:* $\|A\|_\lambda = \sup_{\|u\|_\lambda = 1, \|u\|_\lambda \neq 0} \|Au\|_\lambda$.

Define the Fourier transform of $DP|_0$ by $\mathcal{F}(DP|_0) =: \widehat{DP}|_0(f)$.

*Condition 1:* $|\widehat{DP}|_0(f)| > \beta > 0\ \forall f$ (i.e., no finite or infinite zeros on $j\omega$ axis). Observe that $L_\infty[0,T] \subset L_2$.

*Theorem 1:* If the assumptions ($\mathcal{A}1$–$\mathcal{A}5$) and Condition 1 are satisfied, then the algorithm (3) produces a sequence of inputs which converges to $u^*$, if there are no disturbances (i.e., $w_i = 0$ and $v_i = 0$) and no initialization errors ($x_i(0) = x_d(0)\ \forall i$). If $w_i$ and $v_i$ are bounded and the initial state error is bounded ($\|x_d(0) - x_i(0)\| < b_{x0}$), $u_i$ converges to $B(u^*, r)$, as $i \rightarrow \infty$. The radius $r$ of the ball $B(u^*, r)$ depends continuously on the bounds on the disturbances $w_i$ and $v_i$ and the initialization error. If there exists a $u_d \in L_\infty \cap C_0[0,T]$ with $P(u_d) = y_d$, then $u_i$ converges to the desired input solution $u_d$.

*Proof:* The proof relies on the application of a variant of the contraction mapping theorem [5] to the input sequence. The main idea of the proof is to show that $\|\bar{\delta}u_{i+1}\|_\lambda \leq \rho\|\bar{\delta}u_i\|_\lambda + b_d$ where $0 \leq \rho < 1(\bar{\delta}u_i \triangleq u_d - u_i)$. This implies that $\lim_{i\to\infty}\sup\|\bar{\delta}u_i\|_\lambda \to 1/(1-\rho)b_d$, where $b_d$ is a continuous function of the bounds on disturbances and initialization error. Construct the sequence $\{u_i(\cdot)\}_{i=0}^{\infty}$ by defining:

$$u_0 = \mathcal{T}(u_0); \quad u_{i+1} = T_{(x_i(\cdot),w_i(\cdot))}[u_i(\cdot)]; \\ \triangleq \mathcal{T}\left(u_i + (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*(y_d - P(u_i))\right).$$

$T_{(x_i(\cdot),w_i(\cdot))}[u_i(\cdot)]$ is denoted by $T_i(u_i)$ for simplicity in the rest of this note. Now, from (3) and the linearity of the truncation operator, (12), shown at the bottom of the page, holds. Following [6], we show that the Frechet derivative of $P$ is given by $DP|_{u_i}$. That is, $DP|_{u_i}$ satisfies

$$\lim_{\|\delta u_i\|\to 0} \frac{\|P(u_i+\delta u_i) - P(u_i) - DP|_{u_i}[\delta u_i]\|}{\|\delta u_i\|} = 0 \tag{13}$$

In (13), let $s(\delta u_i)$ be defined by: $s(\delta u_i) \triangleq P(u_i+\delta u_i) - P(u_i) - DP|_{u_i}[\delta u_i]$. From (13), we can see $s$ is $o(\delta u_i)$. Denoting $\delta u_i \triangleq [v_i - u_i]$, we can rewrite (12) as

$$\begin{aligned}\|T_i(u_i) - T_i(v_i)\| &= \Big\|\mathcal{T}\Big(-\delta u_i - (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^* P(u_i) \\ &\quad \cdot + (\alpha I + DP|_0^* DP|_0)^{-1} \\ &\quad \cdot DP|_0^*\left(s(\delta u_i) + P(u_i) + DP|_{u_i}[\delta u_i]\right)\Big)\Big\| \\ &= \Big\|\mathcal{T}\Big(-\delta u_i + (\alpha I + DP|_0^* DP|_0)^{-1} \\ &\quad \cdot DP|_0^*\left(s(\delta u_i) + DP|_{u_i}[\delta u_i]\right)\Big)\Big\|.\end{aligned}$$

Since $s(\delta u_i)$ is $o(\delta u_i)$, $\lim_{\|\delta u_i\|\to 0}(\|(\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*\|\|s(\delta u_i)\|)/\|\delta u_i\| = 0$. This implies that $\forall\epsilon > 0,\ \exists\,\delta > 0$ such that $\|\delta u_i\| \leq \delta$

$$\Rightarrow \frac{\|(\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*\|\|s(\delta u_i)\|}{\|\delta u_i\|} \leq \epsilon < 1. \tag{14}$$

*Bounding $\delta\tilde{x}_i$ and $\delta x$:* From assumptions $\mathcal{A}1$, $\mathcal{A}3$, $\mathcal{A}4$: $\|A_i\| \leq \|f_x(x_i)\| + \|g_x(x_i)\|\|u_i\| \leq b_A$, $\|B_i\| = \|g(x_i)\| \leq b_g$, $\|C_i\| = \|h_x(x_i)\| \leq b_{hx}$, $\|b_i\| \leq b_b$. Therefore, $\|A\| \leq b_A,\ \|B\| \leq b_g,\ \|C\| \leq b_{hx}$. From (6), we can write

$$\delta\tilde{x}_i(t) = \delta\tilde{x}_i(0) + \int_0^t A_i(\tau)\delta\tilde{x}_i(\tau) + B_i(\tau)\delta u_i(\tau) - b_i(\tau)w_i(\tau)d\tau. \tag{15}$$

Therefore, using the triangle-inequality and bounds on $b_i$ and $w_i$, we have $\|\delta\tilde{x}_i\| \leq \|\delta\tilde{x}_i(0)\| + \int_0^t \|A_i(\tau)\|\|\delta\tilde{x}_i(\tau)\| + \|B_i(\tau)\|\|\delta u_i(\tau)\| + b_b b_w d\tau$. Using Gronwall–Bellman inequality (see [5, p. 63]) $\|\delta\tilde{x}_i\| \leq e^{b_A t}\|\delta\tilde{x}_i(0)\| + \int_0^t e^{b_A(t-\tau)}(b_g\|\delta u_i(\tau)\| + b_b b_w)d\tau$. Multiplying (15) by $e^{-\lambda t}$, defining $K_1 \triangleq \max(b_A, b_g)$ and assuming $\lambda > K_1$ we have

$$\begin{aligned}e^{-\lambda t}\|\delta\tilde{x}_i\| \leq\, & e^{(b_A-\lambda)t}\|\delta\tilde{x}_i(0)\| \\ & + K_1\int_0^t e^{(K_1-\lambda)(t-\tau)}e^{-\lambda\tau}\|\delta u_i(\tau)\|\,d\tau \\ & + b_b b_w\int_0^t e^{-\lambda\tau}e^{(b_A-\lambda)(t-\tau)}d\tau.\end{aligned}$$

Note that for a constant $\|k\|_\lambda = k$. Taking $\sup$ over $t \in [0,T]$ we have

$$\begin{aligned}\|\delta\tilde{x}_i\|_\lambda \leq\, & \|\delta\tilde{x}_i(0)\| \\ & + \frac{K_1}{\lambda - K_1}\left(1 - e^{(K_1-\lambda)T}\right)\|\delta u_i(\tau)\|_\lambda \\ & + \frac{b_b b_w}{\lambda - b_A}\left(1 - e^{(b_A-\lambda)T}\right).\end{aligned} \tag{16}$$

Similarly from (4), it can be proved

$$\|\delta x\|_\lambda \leq \|\delta x(0)\| + \frac{K_1}{\lambda - K_1}\left(1 - e^{(K_1-\lambda)T}\right)\|\delta u_i(\tau)\|_\lambda \tag{17}$$

where $\delta u_i$ is the input to (4).

*Defining $\Delta DP|_{u_i}$:* Define a linear operator $\Delta DP|_{u_i} \triangleq DP|_{u_i} - DP|_0$, so that

$$\Delta DP|_{u_i} : \delta u_i \mapsto \Delta y_i; \quad L_\infty \to L_\infty. \tag{18}$$

From (6), the output of the operator $DP|_{u_i}$ is: $\delta\tilde{y}_i(t) = C_i(t)\delta\tilde{x}_i(t) - v_i(t)$ and from (4) the output of the operator $DP|_0$ is $\delta y(t) = C\delta x(t)$. This implies, $\Delta y_i(t) = \delta\tilde{y}_i(t) - \delta y(t) = C_i(t)\delta\tilde{x}_i(t) - v_i(t) - C\delta x(t)$. Therefore, using (16), (17), and the bound on $v_i$ we can write

$$\begin{aligned}\|\Delta y_i\|_\lambda \leq\, & \|C_i\|_\lambda\|\delta\tilde{x}_i\|_\lambda + \|C\|_\lambda\|\delta x\|_\lambda + b_v \\ \leq\, & b_{hx}\left(\|\delta\tilde{x}_i\|_\lambda + \|\delta x\|_\lambda\right) + b_v \\ \leq\, & \frac{2b_{hx}K_1}{\lambda - K_1}\left(1 - e^{(K_1-\lambda)T}\right)\|\delta u_i\|_\lambda \\ & + \frac{b_{hx}b_b b_w}{\lambda - b_A}\left(1 - e^{(b_A-\lambda)T}\right) \\ & + b_{hx}\left(\|\delta\tilde{x}_i(0)\|_\lambda + \|\delta x(0)\|_\lambda\right) + b_v.\end{aligned}$$

*Showing Contraction Mapping:* From (12), we have the equation shown at the bottom of the next page. Define $\|(\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*\|_\lambda = G_1$. It can be shown in the following way that if $DP|_0$ satisfies *Condition 1*, then $\|\alpha(\alpha I + DP|_0^* DP|_0)^{-1}[\delta u_i]\|_\lambda \leq \epsilon_\alpha\|\delta u_i\|_\lambda$, where $0 < \epsilon_\alpha < 1$. With the choice of $\alpha$ sufficiently small, $\epsilon_\alpha$ can be made arbitrarily small.

Let $\bar{y} = DP|_0\bar{u}$, $\bar{Y}(f) := \mathcal{F}\bar{y}(t)$ and $\bar{U}(f) := \mathcal{F}\bar{u}(t)$.

$$\begin{aligned}\|DP|_0\bar{u}\|^2 \triangleq\, & \int_{-\infty}^{\infty}\bar{y}(t)\bar{y}^*(t)dt \\ =\, & \int_{-\infty}^{\infty}\bar{Y}(f)\bar{Y}^*(f)df \\ =\, & \int_{-\infty}^{\infty}[\widehat{DP}|_0(f)\bar{U}(f)][\widehat{DP}|_0(f)\bar{U}(f)]^*df.\end{aligned}$$

($\mathcal{F}$: Fourier Transform)

$$\begin{aligned}\|T_i(u_i) - T_i(v_i)\| &= \Big\|\mathcal{T}\left(u_i + (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*(y_d - P(u_i))\right) \\ &\quad - \mathcal{T}\left(v_i + (\alpha I + DP|_0^* DP|_0)^{-1}\cdot DP|_0^*(y_d - P(v_i))\right)\Big\| \\ &= \Big\|\mathcal{T}\left(u_i - (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^* P(u_i) - v_i + (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^* P(v_i)\right)\Big\|.\end{aligned} \tag{12}$$

If *Condition 1* is satisfied, then $\|DP|_0\bar{u}\|^2 > \beta^2\|\bar{u}\|^2$ where $\beta > 0$. Consider again (19). Let $\delta u_i = (DP|_0^* DP|_0 + \alpha I)[\bar{u}]$ so that $\|(DP|_0^* DP|_0 + \alpha I)^{-1}\delta u_i\|^2 = \|\bar{u}\|^2$. Note that

$$\begin{aligned}\|(DP|_0^* DP|_0 + \alpha I)[\bar{u}]\|^2, & \\ =&\|(DP|_0^* DP|_0)[\bar{u}]\|^2 + 2\alpha\langle DP|_0^* DP|_0\bar{u}, \bar{u}\rangle + \alpha^2\|\bar{u}\|^2 \\ =&\|(DP|_0^* DP|_0)[\bar{u}]\|^2 + 2\alpha\|DP|_0[\bar{u}]\|^2 + \alpha^2\|\bar{u}\|^2 \\ >&(2\alpha\beta^2 + \alpha^2)\|\bar{u}\|^2 \\ \Rightarrow&\|\bar{u}\|^2 < (2\alpha\beta^2 + \alpha^2)^{-1}\|(DP|_0^* DP|_0 + \alpha I)[\bar{u}]\|^2. \quad (19)\end{aligned}$$

Therefore, we can write, $\|\alpha(DP|_0^* DP|_0 + \alpha I)^{-1}[\delta u_i]\|^2 = \alpha^2\|\bar{u}\|^2$; *[using (19)]* $\leq \alpha^2(2\alpha\beta^2 + \alpha^2)^{-1}\|(DP|_0^* DP|_0 + \alpha I)[\bar{u}]\|^2 = \epsilon_\alpha^2\|\delta u_i\|^2$ where $\epsilon_\alpha = [\alpha^2(2\alpha\beta^2 + \alpha^2)^{-1}]^{1/2} < 1$. With the choice of $\alpha$, $\epsilon_\alpha$ can be made arbitrarily small.

If the transfer function corresponding to $DP|_0$ is strictly proper then the *Condition 1* is not satisfied at $f = \infty$. Then as $f \to \infty$ $\epsilon_\alpha(f) \to 1$ and, intuitively, high frequency components of the input sequence would converge slowly. In that case, the learning controller must be modified in the following way:

Instead of considering $(DP|_0^* DP|_0 + \alpha I) DP|_0^*$ as the learning operator, take $(\widetilde{DP}|_0^* \widetilde{DP}|_0 + \alpha I)\widetilde{DP}|_0^*$ as the modified learning controller, where $\widetilde{DP}|_0 = (DP|_0 + \epsilon_D)$ is obtained by adding a feedforward term to $DP|_0$. Hence, $\widetilde{DP}|_0$ is given by modified (4) as follows:

$$\begin{aligned}\dot{\delta x}(t) =& A\delta x(t) + B\delta u(t), \quad \delta x(\pm\infty) = 0 \\ \delta y(t) =& C\delta x(t) + \epsilon_D\delta u(t) \quad (20)\end{aligned}$$

where $0 < \epsilon_D < 1$. The modified operator satisfies *Condition 1* and the convergence analysis proceeds in the same way with $\epsilon_D$ sufficiently small. Substituting the bounds on $\|\Delta y_i\|_\lambda = \|\Delta DP|_{u_i}[\delta u_i]\|_\lambda$ from (19) and multiplying (19) by $e^{-\lambda t}$, we can write $\lambda$—*norm* of (19) taking $\sup$ over $t \in [0, T]$ as

$$\begin{aligned}\|T_i(u_i) - &T_i(v_i)\|_\lambda \\ \leq& G_1\|\Delta y_i\|_\lambda + \epsilon_\alpha\|\delta u_i\|_\lambda + \epsilon\|\delta u_i\|_\lambda \\ \leq& G_1\left\{\frac{2b_{hx}K_1}{\lambda - K_1}\left(1 - e^{(K_1-\lambda)T}\right)\|\delta u_i\|_\lambda \right. \\ &+ \frac{b_{hx}b_b b_w}{\lambda - b_A}\left(1 - e^{(b_A-\lambda)T}\right) \\ &\left. + b_{hx}(\|\delta\tilde{x}_i(0)\|_\lambda + \|\delta x(0)\|_\lambda) + b_v\right\} \\ &+ \epsilon_\alpha\|\delta u_i\|_\lambda + \epsilon\|\delta u_i\|_\lambda \\ =& \underbrace{\left[G_1\left\{\frac{2b_{hx}K_1}{\lambda - K_1}\left(1 - e^{(K_1-\lambda)T}\right)\right\} + \epsilon_\alpha + \epsilon\right]}_{\rho_i} \\ &\cdot\|\delta u_i\|_\lambda + K_w b_w + K_v b_v + K_0 \\ \leq& \rho\|\delta u_i\|_\lambda + b_d\end{aligned}$$

where $K_0$ is the norm bounds of the initial state errors. $K_w b_w$ and $K_v b_v$ are the norm bounds of the input and output disturbances respectively. Since $\epsilon < 1$, with sufficiently small $\alpha$, we can find a $\lambda > K_2 \geq K_1$ which makes $\rho_i \leq \rho < 1$. Therefore, we can write: $\|u_{i+1} - v_{i+1}\|_\lambda \leq \rho_i\|u_i - v_i\|_\lambda + b_d \leq \rho_i\rho_{i-1}\|u_{i-1} - v_{i-1}\|_\lambda + \rho_i b_d + b_d \leq \rho^2\|u_{i-1} - v_{i-1}\|_\lambda + \rho b_d + b_d$ where $b_d$ combines the norm bounds of the initial state errors of the controller and disturbances. Therefore, $\limsup_{i\to\infty}\|v_{i+1} - u_{i+1}\|_\lambda \leq 1/(1-\rho)b_d$; i.e., $\exists N$ such that $\forall i > N$, $u_i \in B(u^*, r)$, where $u^*$ is the fixed point of the contraction mapping $T_i$ and $B(u^*, r)$ is an open ball of radius $r = 1/(1-\rho)b_d$ and center $u^*$. If the disturbances and initialization errors are absent, $b_d = 0$, and hence, $u_i$ converges to $u^*$. If $\exists\, u_d$ such that $P(u_d) = y_d$, the fixed point $u^*$ of the contraction mapping $T_i(\cdot)$ is shown to be $u_d$ in the absence of $w_i$, $v_i$ and initialization error. If $u_i = u_d$, $y_i = y_d$ and $\delta y_i = y_d - y_i = 0$. This implies, the output $(\delta u_i)$ of the learning controller $(\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*$ is zero. Therefore, $T_i(u_d) = u_d + \mathcal{T}((\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*(y_d - y_d)) = u_d + 0 = u_d$. $\Rightarrow u^* = u_d$.

Once the contraction is proved, it can be shown that $T_i$ (as previously defined) is also a mapping from a closed subspace of the Banach space $(L_\infty[0, T])$ into itself, and hence, $T_i$ is a contraction mapping. To show this, consider a desired trajectory $y_d \equiv 0$. Then from (2), $u_d \equiv 0$ for $x_d(0) = 0$. In (12), if we consider $v_i = u_d = 0$, then, since $\|T_i(u_i)\| < \rho\|u_i\|$ (where $\rho < 1$), $T_i$ is a mapping from a closed ball around $u_d \equiv 0$ into itself. Note that, the size of the ball around $u_d \equiv 0$ must be small enough that (14) is also satisfied. Hence, if the initial trajectory $u_0$ lies in the neighborhood of $u_d = 0$, $T_i$ maps the neighborhood into itself for all $i$. Without loss of generality, we can consider another pair $yd = \hat{y}_d$ and $u_d = \hat{u}_d$ (as given by (2). By continuity, $T_i$ maps a closed neighborhood into itself even when $y_d$ is sufficiently close to $\hat{y}_d$. This is the motivation of $\mathcal{A}5$.

## III. Simulation Results

### *Simulation Results With Input Disturbances*

In this section, we perform simulation studies with a single-input–single-output (SISO) nonlinear nonminimum phase plant $P$, *stable-in-first-approximation*, and also input-to-state stable, with input disturbance described by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} = \underbrace{\begin{bmatrix}-x_1 + x_2\\ -3x_2 + x_1^3\\ x_1 - 2x_3\\ -x_4 + x_3^2\end{bmatrix}}_{f(\mathbf{x})} + \underbrace{\begin{bmatrix}0.2 + 0.1\sin^2(x_4)\\ 0\\ 0\\ 0\end{bmatrix}}_{g(\mathbf{x})} u_i + \underbrace{\begin{bmatrix}0.2\sin(40x_1)\\ 0\\ 0\\ 0\end{bmatrix}}_{b(\mathbf{x})} w_i$$

$$y_i(t) = \underbrace{x_1(t) - 3x_3(t)}_{h(\mathbf{x})} + v_i(t); \quad \mathbf{x}(\pm\infty) = 0. \quad (21)$$

$$\begin{aligned}\|T_i(u_i) - T_i(v_i)\| =& \|\mathcal{T}(-\delta u_i + (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*(s(\delta u_i) + DP|_{u_i}[\delta u_i]))\| \\ \leq& \|\mathcal{T}(-\delta u_i + (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^* DP|_{u_i}[\delta u_i]))\| + \epsilon\|\mathcal{T}(\delta u_i)\| \text{ from } (14) \\ & \quad substituting\ DP|_{u_i} = DP|_0 + \Delta DP|_{u_i} \\ =& \|\mathcal{T}(-\delta u_i + (\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*(DP|_0 + \Delta DP|_{u_i})[\delta u_i]))\| + \epsilon\|\mathcal{T}(\delta u_i)\| \\ =& \|\mathcal{T}(-\delta u_i + (\alpha I + DP|_0^* DP|_0)^{-1}\{(DP|_0^* DP|_0 + \alpha I) + (DP|_0^*\Delta DP|_{u_i} - \alpha I)\}[\delta u_i]))\| + \epsilon\|\mathcal{T}(\delta u_i)\| \\ \leq& \|\mathcal{T}((\alpha I + DP|_0^* DP|_0)^{-1} DP|_0^*\Delta DP|_{u_i}[\delta u_i] - \alpha(\alpha I + DP|_0^* DP|_0)^{-1}[\delta u_i])\| + \epsilon\|\mathcal{T}(\delta u_i)\|.\end{aligned}$$

First, we consider the output disturbance $v_i(t)$ to be absent. The reference output trajectory is given by: $y_d(t) = 0.2\sin(t)\ t \in [0, 2\pi]; 0,$ otherwise.

$DP|_0$ is defined by linearizing the system (21) as

$$\dot{\delta x}(t) = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A} \delta x(t) + \underbrace{\begin{bmatrix} 0.2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} \delta u(t)$$

$$\delta y(t) = \underbrace{\begin{bmatrix} 1 & 0 & -3 & 0 \end{bmatrix}}_{C} \delta x(t), \quad \delta x(\pm\infty) = 0.$$

Since the linear controller is unstable, we apply noncausal stable solution approach [2]. We introduce $w_i$ as a bounded input disturbance. $w_i$ is normally distributed random numbers bounded between $\pm 1$. Matlab simulation [Fig. 2(a) and (b)] shows near perfect tracking of the desired output trajectory after a couple of iterations. Note that the remaining error resulting from slow convergence of high frequency components.

*Simulation Results With Input and Output Disturbances*

Now, we introduce $v_i$ as a random bounded output disturbance to the same nonlinear system given by (21). Input disturbance $w_i$ as introduced earlier is also present. Matlab simulation (Fig. 3) shows good tracking of the desired output trajectory after three iterations.

*A. Discussion*

This ILC scheme has some advantages over that presented in [1]. In [1], the inverse of the linearized plant, $DP|_0^{-1}$ is taken to be the learning operator. This necessitates taking the derivative of the output to invert the system. In practice, derivatives cannot be reliably computed in the presence of output sensor noise. Furthermore, the plant may itself produce an output signal that is not differentiable. In this new learning algorithm, however, it is not necessary to take the derivative of the output in order to calculate the update term of the system input at every iteration. (Note that $\alpha$ should be nonzero).

The frequency responses of the linearized plant $DP|_0$ [as given by (4)] and its exact inverse $DP|_0^{-1}$ and pseudo-inverse $DP|_0^{\dagger,\alpha}$ (with $\alpha = 0.001$) are shown in Figs. 4(a) and (b). In our previous scheme [1], the learning operator $DP|_0^{-1}$ has high gain at high frequency as shown in Fig. 4(b). Therefore, the high frequency noise is amplified by the learning operator. From Fig. 4(b) we see that the frequency response of $DP|_0^{\dagger,\alpha}$ behaves similarly to $DP|_0^{-1}$ at low frequencies, but rolls off at high frequencies demonstrating a lowpass nature. Thus the high frequency sensor noise is filtered out. The phase responses of the exact inverse and the pseudo-inverse are identical (see Fig. 4(b). Note that $(\alpha I + DP|_0^{*} DP|_0)$ is a *zero phase filter*. Excellent tracking of the low frequency components is achieved after a few iterations, while the high frequency components of the output error signal converge more slowly. This behavior is corroborated by Fig. 2 (a) and (b), where we see that the low frequency error converges to zero within the first few iterations, while the high frequency error spikes take a larger number of iterations to decay.

In [7], the scale factor is the scalar "gamma," and in our note, the weight is an operator (not necessarily causal) given through the construction of a regularized pseudo inverse. Interestingly, in both schemes the phase of the learning controller is equal to the negative of the plant phase. Our note builds on the earlier work in that the operator weight produces an inverse of the plant over a bandwidth and one can expect rapid convergence in that frequency band. Further, if a multivariable plant has a significant spread between its minimum and maximum singular values, the pseudoinverse automatically scales the learning controller gains along the different spatial directions of the plant. Furuta and Yamakita's delta-modified steepest descent method [7] shares the same high-frequeny roll-off characteristics as the pseudoinverse learning controller. Also, in [8] and [9], we find the application of inverting the transfer function for a robot feedback control system to design the controller and cut off the learning as needed.

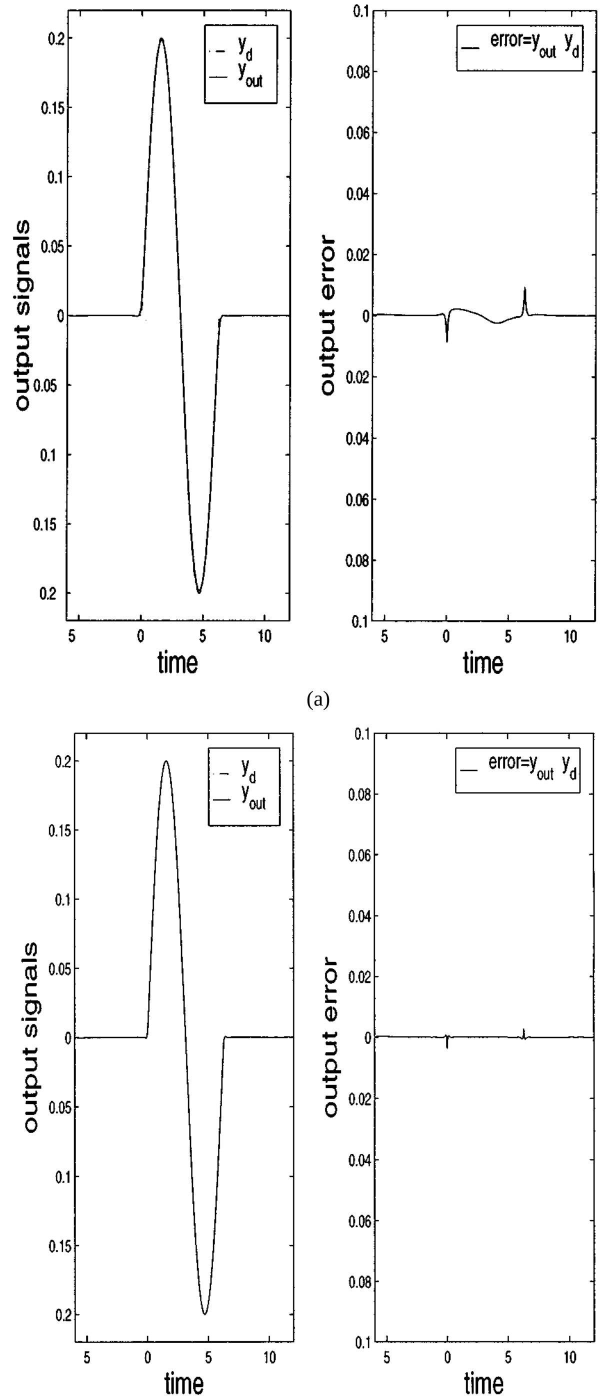



Fig. 2. Tracking of nonlinear nonminimum phase system with input disturbance (a) after three iterations and (b) after 10 iterations.

## IV. Conclusion

The learning algorithm presented in this note guarantees learning, under quite general assumptions. Theoritical assertions are corrobo-

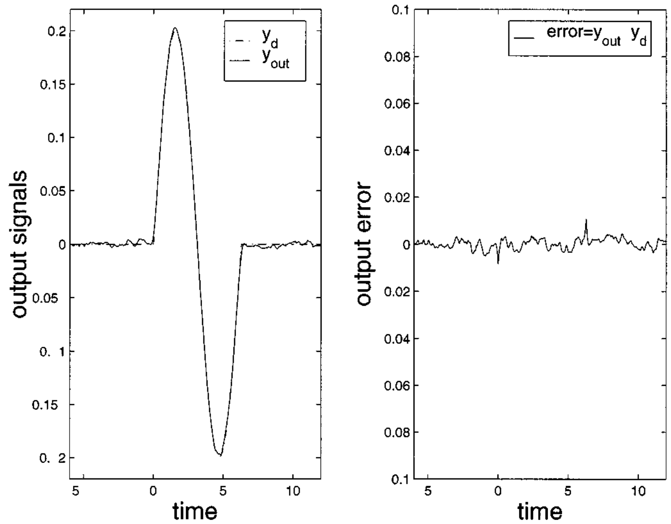



Fig. 3. Tracking of nonlinear nonminimum phase system with input and output disturbances after three iterations. $y_{out}$ is the actual (not measured) output.

rated by simulation results which demonstrate that in the presence of random bounded disturbances the tracking error is uniformly bounded. A major advantage of this scheme is that we are able to eliminate the differentiation operator from the learning update law, which allows us to consider a more general class of nonlinear plants. The learning algorithm can be easily extended to slowly time-varying plants by applying Coppel's method [10]. Learning algorithm is applied to linear plants with unmodeled dynamics in [11] which can be extended to time-varying plants in the future.

## APPENDIX

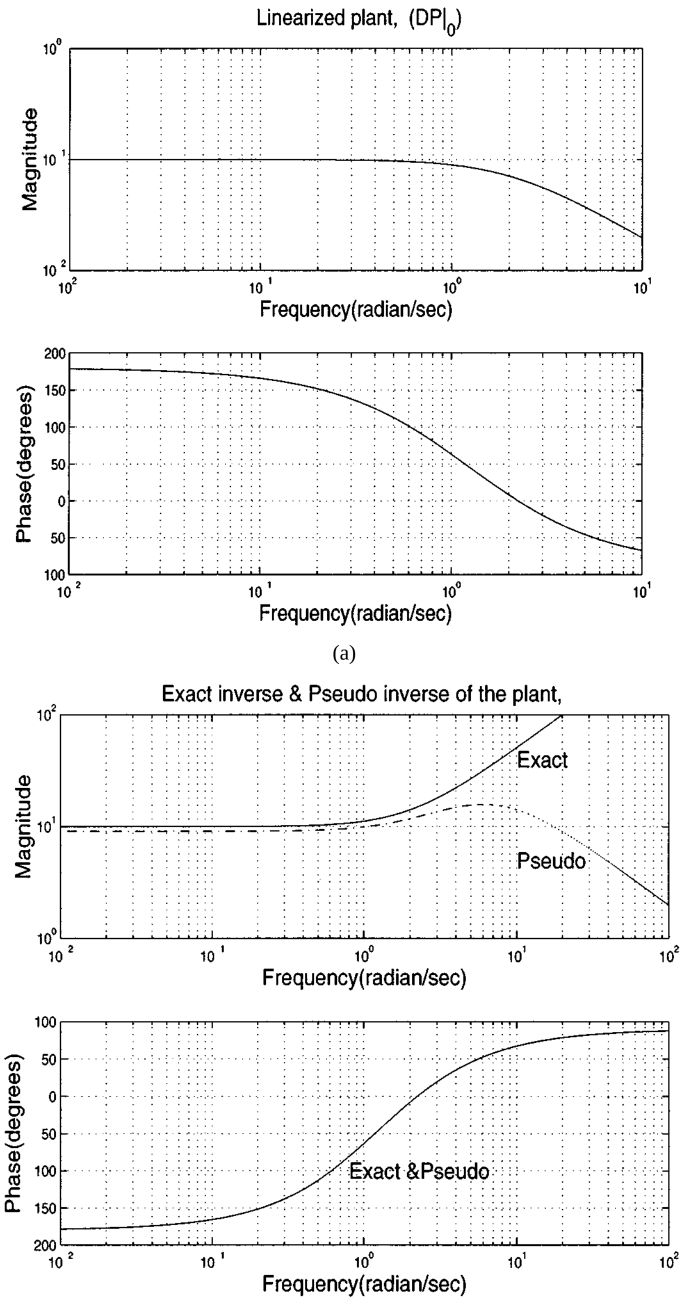
### A. Boundary Value Problem for Nonminimum Phase Systems

A nonlinear nonminimum phase system can be viewed as a mapping of $C_{[0,\infty]}$ to $C_{[0,\infty]}$ or as a mapping from $L_{[-\infty,\infty]}$ to $L_{[-\infty,\infty]}$. In the first case the inverse mapping is unbounded, while in the second, it is bounded but noncausal. It is the second view that enables a proper perspective on tracking control problems as feedforward need not be computed causally from sensor outputs.

If a nonlinear plant with hyperbolic zero dynamics is nonminimum phase, the inverse of the linearized plant is unstable. Hence, we perform stable noncausal inversion of the linearized plant to obtain the learning controller for ILC scheme described in Section III. An essential element to solving this problem is finding a solution meeting boundary conditions at $\pm\infty$. Hence, for the linear learning controller, this reduces to finding solutions to

$$\dot{x} = Ax + Bu, \quad x(\pm\infty) = 0. \tag{22}$$

where we assume that $A$ has no $j\omega$-axis eigenvalues and $u \in L_1 \cap L_\infty$. Without loss of generality, assume that $A$ is block diagonal: $A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}$ where $A_-$ and $-A_+$ are both Hurwitz. It is easily verifiable by substitution into (22) (using the notation $1(t)$ for the unit step function), a bounded state-transition matrix can be defined by: $\phi(t) = \begin{bmatrix} 1(t)e^{A_- t} & 0 \\ 0 & -1(-t)e^{A_+ t} \end{bmatrix}$ and that the solution to (22) meeting the boundary conditions $x(\pm\infty) = 0$ has the form $x(t) = \int_{-\infty}^{+\infty} \phi(t-\tau)Bu(\tau)d\tau$. Define a mapping $\mathcal{A} : L_1 \cap L_\infty \longrightarrow L_1 \cap L_\infty$ as the mapping taking $u$ to $x$ in (22). That is $\mathcal{A} : u \longmapsto x$.




(b)

Fig. 4. Frequency responses (a) $DP|_0$ (b) $DP|_0^{-1}$ and $DP|_0^{\dagger,\alpha}$ (with $\alpha = 0.001$).

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# Frequency-Domain Synthesis of a Filter Using Viète Root Functions

Alain Oustaloup, Patrick Lanusse, and François Levron


***Abstract*—This note proposes a noniterative method to fit a rational transfer function to a specified frequency response. A method using a smoothing of width-modulated pulses of the phase asymptotic-diagram is modified to provide an exact algebraic method. This is then used to synthesize a robust controller and a nonrational transfer function with a fractional differentiation order.**


***Index Terms*—Approximation, filter synthesis, fractional differentiation, frequency-domain method, model reduction, system identification, Viète root functions.**

## I. Introduction

Estimation of the parameters of a dynamic model is a common problem and many methods have been proposed [1]. Most of these methods use sampled-time data to determine the parameters of z-domain transfer functions. Others use frequency-domain data to determine the parameters of s-domain transfer functions. The appropriate type must be chosen for a given problem. This is true for system identification [2].

For a problem such as the synthesis of a filter from its desired frequency response, only a frequency-domain method can be chosen. In automatic control, a filter must be synthesized when determining the rational transfer function of a controller where only the optimal frequency response is known. The use of a frequency-domain method can also be useful for approximating the frequency response of a fully known controller whose order needs to be reduced without reducing performance of the control system.

The first published frequency-domain methods try to minimize a weighted least-squares type cost functions to minimize the difference between the magnitude or the complex value of a desired frequency response and that computed from a transfer function model. These cost functions are generally nonlinear functions of the parameters to find, so their minimization process is very sensitive to the common problem of local minima. To make them linear, Levy proposed in 1959 to multiply the cost function by the unknown denominator of the model [3]. Several variations of the Levy method have been proposed to improve its convergence. The "Frequency domain system identification toolbox" for use with Matlab is one of these variations [2]. In fact, all these methods are based on $l_2$ optimization. Methods based on $l_\infty$ optimization have been presented recently [4], [5]. In 1995, Mathieu proposed a variation [6] of both Levy's and Hakvoort's methods. It consists in canceling the $l_\infty$-norm of Levy's error equation by using a linear programming technique: the simplex method.


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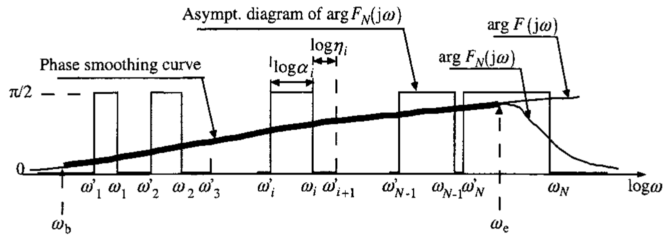


Fig. 1. Fitting of a phase curve from a pulse width modulation.

These synthesis methods, although improved, still use an optimization process and remain sensitive to the local minima problem. Thus, they do not always provide a good fitting of frequency responses. On the contrary, the synthesis method proposed is this note is an original exact algebraic method using mainly the fitting of the phase response data of the frequency response and requiring no optimization process. The procedure is divided into two main steps. First, a linear system is solved whose unknowns are the elementary symmetric functions of roots defined by $F$. Viète (1540–1603). These functions express the relation between the roots of a polynomial and its coefficients. The parameters of the model are then determined from the roots of a polynomial built from these Viète functions and sometimes also using the magnitude data.

Section II presents an initial method using a smoothing of width-modulated pulses of a phase asymptotic-diagram. This is then improved to provide an exact method.

In Section III, the exact method is used to synthesize a nonconservative robust controller to achieve an open-loop frequency response with a constant phase and then a nonrational transfer function with a fractional differentiation order.

## II. Synthesis Method

The synthesis of the transfer function of a filter $F$ consists in interpolating its known sampled frequency response, $F(\mathrm{j}\omega)$. It is achieved by the determination of the parameters of a rational transfer function model $F_N(s)$, which approximates the ideal rational transfer function

$$F(s) = \frac{C_0}{s^q} \frac{\prod_{i=1}^{m} \left(1 + \frac{s}{\omega'_i}\right)}{\prod_{i=1}^{n} \left(1 + \frac{s}{\omega_i}\right)} \tag{1}$$

where the number of integrators, $q$ and the gain, $C_0$, can easily be determined and where $\omega'_i$ and $\omega_i$ are corner frequencies which correspond to real or complex zeros and poles (their opposite values).

### A. Initial Method

The initial method aims to fit, over a frequency range $[\omega_b, \omega_e]$, the known phase curve $\arg F(\mathrm{j}\omega)$ of the filter to be synthesized, with the phase of rational transfer function $F_N(s)$ containing an equal number