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An algorithm for the determination of optimal cutting patterns

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Abstract

This paper presents a new mathematical programming formulation for the problem of determining the optimal manner in which several product rolls of given sizes are to be cut out of raw rolls of one or more standard types. The objective is to perform this task so as to maximize the profit taking account of the revenue from the sales, the costs of the original rolls, the costs of changing the cutting pattern and the costs of disposal of the trim. A mixed integer linear programming (MILP) model is proposed which is solved to global optimality using standard techniques. A number of example problems, including an industrial case study, are presented to illustrate the efficiency and applicability of the proposed model.

Scope and purpose

One-dimensional cutting stock (trim loss) problems arise when production items must be physically divided into pieces with a diversity of sizes in one dimension (e.g. when slitting master rolls of paper into narrower width rolls). Such problems occur when there are no economies of scale associated with the production of the larger raw (master) rolls. In general, the objectives in solving such problems are to [5]:

- minimize trim loss;
- avoid production over-runs and/or;
- avoid unnecessary slitter setups.

The above problem is particularly important in the paper converting industry when a set of paper rolls need to be cut from raw paper rolls. Since the width of a product is fully independent of the width of the raw paper a highly combinatorial problem arises. In general, the cutting process always produces inevitable trim-loss which has to be burned or processed in some waste treatment plant. Trim-loss problems in the paper industry have, in recent years, mainly been solved using heuristic rules. The practical problem formulation has, therefore, in most cases been restricted by the fact that the solution methods ought to be able to handle the entire problem. Consequently, only a suboptimal solution to the original problem has been obtained and

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very often this rather significant economic problem has been left to a manual stage. This work presents a novel algorithm for efficiently determining optimal cutting patterns in the paper converting process. A mixed-integer linear programming model is proposed which is solved to global optimality using available computer tools. A number of example problems including an industrial case study are presented to illustrate the applicability of the proposed algorithm.

Keywords: Integer programming; Optimization; Trim-loss problems; Paper converting industry

1. Introduction

An important problem which is frequently encountered in industries such as paper is related with the most economic manner in which several product roll of given sizes are to be produced by cutting one or more wider raw rolls available in one or more standard widths. The solution of this problem involves several interacting decisions:

- The number of product rolls of each size to be produced.
  This may be allowed to vary between given lower and upper bounds. The former normally reflect the firm orders that are currently outstanding, while the latter correspond to the maximum capacity of the market. However, certain discounts may have to be offered to sell sheets over and above the quantities for which firm orders are available.
- The number of raw rolls of each standard width to be cut.
  Rolls may be available in one or more standard widths, each of a different unit price.
- The cutting pattern for each raw roll.
  Cutting takes place on a machine employing a number of knives operating in parallel on a roll of standard width. While the position of the knives may be changed from one roll to the next, such changes may incur certain costs. Furthermore, there may be certain technological limitations on the knife positions that may be realized by any given cutting machine.

The optimal solution of the above problem is often associated with the minimization of the "trim" waste that is generally unavoidable since rolls of standard widths are used. However, trim-loss minimization does not necessarily imply minimization of the cost of the raw materials (rolls) being used especially if several standard roll sizes are available. A more direct economic criterion is the maximization of the profit of the operation taking account of:

- the revenue from product rolls sales, including the effects of any bulk discounts;
- the cost of the rolls that are actually used;
- the costs, if any, of changing the knife positions on the cutting machine;
- the cost of disposing of trim waste.

The above constitutes a highly combinatorial problem and it is not surprising that traditionally its solution has often been carried out manually based on human expertise. The simplified version of this problem is similar to the cutting stock problem known in the operation-research literature, where a number of ordered pieces need to be cut off bigger stored pieces in the most economic fashion. In the 1960s and the 1970s, several scientific articles were published on the problem of minimizing trim loss, e.g. [1,2]. Hinxman [3] presents a good overview of the available solution methods for trim-loss and assortment problems.

Gilmore and Gomory [1] presented a basic linear programming approach to the cutting stock problem while relaxing some integer-characters of the problem. Gilmore and Gomory [2] described an iterative solution method that is suitable for very large number of orders and is computational cheap, but the resulting values for the number of cutting patterns to be used are non-integer and it is not possible to prove the optimality or indicate the margin of optimality of these cutting patterns. Thus, the rounding values obtained by the algorithm of Gilmore and Gomory [2] may very probably result in poor economic performance. Wascher [4] presented a linear programming approach to cutting stock problems taking into account multiple objectives such as cost of the raw materials, cost of the overproduction storage, trim-loss removal costs, etc. Sweeny [5] proposed a heuristic procedure for solving one-dimensional cutting stock problems with multiple quality grades. Ferreira et al. [6] considered the two-phase roll cutting problems based on a heuristic approach. Gradisar et al. [7] presented an efficient sequential heuristic procedure and a software tool for optimization of roll cutting in the clothing industry. Later, Gradisar et al. [8] developed an improved solution strategy based on a combination of approximations and heuristics leading to almost optimal solutions for the one-dimensional stock cutting problems. A software tool was also developed.

In recent years, integer programming techniques have been used for the solution of the trim-loss and production optimization problem in the paper industry. The work of Westerlund and coworkers at Åbo Akademi University in Finland is a key contribution in this area. Harjunkoski [9] considered a mathematical programming approach to the trim-loss problem and presented two different types of formulation. In the first one, both the cutting patterns that need to be used and the number of rolls that have to be cut according to each such pattern are treated as unknowns. This results in an integer nonlinear mathematical problem (INLP) involving bilinear problems of the variables characterizing each cutting pattern and the corresponding number of rolls cut in this way. Two different ways of linearizing the INLP to obtain a mixed integer linear programming (MILP) mode were presented. However, these linearizations often result in a significant increase in the number of variables and constraints, as well as a large integrality gap. The second type of formulation presented by Harjunkoski [9] is based on using a fixed set of cutting patterns that is decided a priori. This results in a MILP that has a much smaller integrality gap than the one resulting from the linearization of the INLP formulation mentioned above. However, the solution obtained is guaranteed to be optimal only if all non-inferior cutting patterns are identified and taken into consideration. The number of such patterns may be quite substantial for realistic industrial problems. Extending the above work, Harjunkoski [10] presented linear and convex formulations for solving the non-convex trim-loss problems.

Westerlund [11] considered the two-dimensional trim-loss problem in paper converting. A non-convex optimization model was proposed where both the widths and the lengths of the raw paper were considered as variables. A two-step solution procedure was used where all feasible cutting patterns were first generated and then a MILP problem was solved. In a similar fashion, the production optimization problem in the paper converting industry was addressed by Westerlund [12]. Scheduling aspects of the cutting machines in paper converting were simultaneously considered with the trim-loss problem by Westerlund [13]. Recently, Harjunkoski [14] incorporated environmental impact considerations into a general framework for trim-loss minimization.

This paper presents an alternative mathematical programming formulation that results directly in a MILP of small integrality gap. The salient feature of this model is that it does not require a priori enumeration of all possible cutting patterns. The next section presents a formal statement of the problem under consideration and the notation used. Section 3 considers the mathematical formulation of the objective function and the operational constraints. This is followed by some example problems including an industrial case study illustrating the applicability and computational behavior of the proposed formulation.

2. Problem statement and data

The task being considered here is to produce product rolls of I different types, the width of type i being denoted by B_i, $i = 1, \ldots, I$ from one or more standard rolls. The *lengths* of all raw rolls and of the product rolls resulting from them are assumed to be identical and fixed. It is beyond the scope of this work to consider the two-dimensional problem where both the widths and the lengths of raw paper rolls and the cutting patterns are considered variables.

Product rolls are mostly produced to order. The minimum ordered quantity for product rolls of width i is denoted by $N_i^{\min}$ and is given, and so is the corresponding unit price p_i. However, customers may be willing to buy additional rolls of type i up to a maximum quantity $N_i^{\max}$ subject to a discount of c_i^{disc} for each product roll over and above the minimum number $N_i^{\min}$. In general, the number of additional rolls sold in this manner tends to be rather small since the main incentive of such discounting from the point of view of the manufacturer is merely to decrease the loss through trim.

The product rolls are to be cut from raw rolls of T different standard types. The unit price for a raw roll of type t is denoted by c_t^{roll} and its nominal width by B_t^{roll}. However, the useful width of a roll of type t is determined by the cutting machine used. In particular, each raw roll type $t = 1, \ldots, T$ is characterized by a maximum possible total engagement $B_t^{\max}$ denoting the maximum total width of all product rolls that can be cut from a raw roll of this type. There may also be a minimum required total engagement $B_t^{\min}$ for this type of roll. In general

$$B_t^{\min} \leqslant B_t^{\max} \leqslant B_t, \quad \forall t = 1, \ldots, T.$$

The maximum number $N_t^{\max}$ of product rolls that can be cut out of a raw roll of type t will generally be determined by the knives and other characteristics of the available machine. Moreover, in some cases, there may be limitations in the available number J_t^* of raw rolls of a given type t.

The cutting pattern for each raw roll is determined by the position of the knives. Frequent changes in these positions are generally undesirable. Each such change may therefore be associated with a non-zero cost c^{change}.

The production of the required product rolls from the available raw rolls may result in trim waste which may need to be disposed of. The cost of such disposal per unit width of trim is denoted by c^{disp}.

Based on the given data, we first derive an upper bound J on the number of raw rolls that may need to be cut. This is obtained by assuming that the maximum number $N_i^{\max}$ of product rolls of each type i will be produced; that raw rolls of the type t that permits the smallest minimum engagement $B_t^{\min}$ will be used; and that each raw roll will be used to produce product rolls of a single type only. Overall, this leads to the following upper bound on the number of raw rolls that may be required:

$$J^{\max} = \sum_{i=1}^{I} \left\lceil \frac{N_i^{\max}}{\lfloor \min_t B_t^{\min}/B_i \rfloor} \right\rceil. \tag{1}$$

We can also calculate a lower bound $J^{\min}$ on the minimum number of raw rolls that are necessary to satisfy the minimum demand for the existing orders. We do this by assuming that rolls of the type t allowing the maximum possible engagement $B_t^{\max}$ are used, and that no trim is produced. However, we must also take account of possible limitations on the number of available knives. Overall, this leads to the following lower bound on the number of rolls that may be required:

$$J^{\min} = \max\left\{ \left\lceil \frac{\sum_{i=1}^{I} N_i^{\min} B_i}{\max_t B_t^{\max}} \right\rceil, \quad \left\lceil \frac{\sum_{i=1}^{I} N_i^{\min}}{\max_t N_t^{\max}} \right\rceil \right\}. \tag{2}$$

3. Mathematical formulation

The aim of the mathematical formulation is to determine the type t of each raw roll j to be cut and the number of product rolls of each type i to be produced from it.

3.1. Key variables

The following integer variables are introduced:

n_{ij}: number of product rolls of type i to be cut out of raw roll j
Δ_i: number of product rolls of type i produced over and above the minimum number ordered.

We note that n_{ij} cannot exceed:

- the maximum number $N_i^{\max}$ of product rolls of type i that can be sold;
- the maximum number of product rolls of width B_i that can be accommodated within a maximum engagement $B_t^{\max}$ for a raw roll of type t;
- the maximum number $N_t^{\max}$ of knives that can be applied to a raw roll of type t.

This leads to the following bounds for n_{ij}:

$$0 \leqslant n_{ij} \leqslant \min\left(N_i^{\max}, \max_{1 \leqslant t \leqslant T} \left\lfloor \frac{B_t^{\max}}{B_i} \right\rfloor, \max_{1 \leqslant t \leqslant T} N_t^{\max} \right) \quad \forall i = 1, \ldots, I,\ j = 1, \ldots, J^{\max}. \tag{3}$$

Also

$$0 \leqslant \Delta_i \leqslant N_i^{\max} - N_i^{\min}, \quad \forall i = 1, \ldots, I. \tag{4}$$

We note that Δ_i need to be included in the model only if $N_i^{\max} > N_i^{\min}$. We also introduce the following binary variables:

$$y_{tj} = \begin{cases} 1 & \text{if the } j\text{th roll to be cut is of type } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if the cutting pattern for paper roll } j \text{ is different to that for roll } j-1, \\ 0 & \text{otherwise.} \end{cases}$$

We note that, in general, the index j will be in the range $[1, \ldots, J^{\max}]$. However, the formulation to be presented will assign a type t only to the raw rolls j that are actually used. Hence, the total number of rolls to be cut will also be determined by the solution of the optimization problem. This will become clearer in the next subsection.

3.2. Roll type determination constraints

Each raw roll j to be cut must be of a unique type t. This results in the following constraints:

$$\sum_{t=1}^{T} y_{tj} = 1, \quad \forall j = 1, \ldots, J^{\min}, \tag{5a}$$

$$\sum_{t=1}^{T} y_{tj} \leqslant 1, \quad \forall j = J^{\min} + 1, \ldots, J^{\max}. \tag{5b}$$

Note that for $j > J^{\min}$, it is possible that $y_{tj} = 0$ for *all* types t; this simply implies that it is not necessary to cut roll j.

Furthermore, the limited availability of raw rolls of a given type t may be expressed in terms of the constraint

$$\sum_{j=1}^{J^{\max}} y_{tj} \leqslant J_t^*, \quad \forall t = 1, \ldots, T. \tag{6}$$

3.3. Cutting constraints

We need to ensure that, if a roll j is to be cut, then the limitations on the minimum and maximum engagement are observed. This is achieved via the constraints

$$\sum_{t=1}^{T} B_t^{\min} y_{tj} \leqslant \sum_{i=1}^{I} B_i n_{ij} \leqslant \sum_{t=1}^{T} B_t^{\max} y_{tj}, \quad \forall j = 1, \ldots, J^{\max}. \tag{7}$$

We note that the quantity $\sum_{i=1}^{I} B_i n_{ij}$ represents the total width of all product rolls to be cut out of raw roll j. If $y_{tj} = 1$ for some roll type t, then constraint (7) ensures that

$$B_t^{\min} \leqslant \sum_{i=1}^{I} B_i n_{ij} \leqslant B_t^{\max}.$$

On the other hand, if $y_{tj} = 0$ for *all* t, then constraint (7) effectively forces $n_{ij} = 0$ for all $i = 1, \ldots, I$. This expresses the obvious fact that if roll j is not actually cut, then no product rolls of any type can be produced from it.

We also need to ensure that the number of product rolls cut out of any roll j of type t does not exceed the number of knives that can be deployed on rolls of this type. This is written as

$$0 \leqslant \sum_{i=1}^{I} n_{ij} \leqslant \sum_{t=1}^{T} N_t^{\max} y_{tj}, \quad \forall j = 1, \ldots, J^{\max}. \tag{8}$$

3.4. Production constraints

The total number of product rolls of each type i that are produced comprises the minimum ordered quantity $N_i^{\min}$ for this type plus the surplus production Δ_i:

$$\sum_{j=1}^{J^{\max}} n_{ij} = N_i^{\min} + \Delta_i, \quad \forall i = 1, \ldots, I. \tag{9}$$

These constraints, together with the bounds on Δ_i, ensure that the quantity of product rolls of type i produced lies between the minimum and maximum bounds $N_i^{\min}$ and $N_i^{\max}$, respectively.

3.5. Changeover constraints

If changing the cutting pattern incurs a non-zero cost $c^{\text{change}} > 0$, we need to determine when such changes will take place. To this end, we include the following constraint:

$$-M_i z_j \leqslant n_{ij} - n_{i,j-1} \leqslant M_i z_j, \quad \forall i = 1, \ldots, I,\ j = 2, \ldots, J^{\max}. \tag{10}$$

Note that this will allow z_j to be zero only if $n_{ij} = n_{i,j-1}$ for *all* product rolls i, i.e. if rolls j and $j-1$ are cut in exactly the same way. Here, the constant M_i is an upper bound on n_{ij} (see Section 3.1).

3.6. Objective function

The objective of the optimization is to maximize the total profit of the operation taking account of:

- The income from the sales of product rolls of each type i.
  This comprises the income from selling the minimum ordered quantities $N_i^{\min}$ at the full unit price p_i, plus the income of selling the additional quantities Δ_i at the discounted unit price $p_i - c_i^{\text{disc}}$:

  $$\sum_{i=1}^{I} (p_i N_i^{\min} + \Delta_i (p_i - c_i^{\text{disc}})).$$

- The costs of the rolls to be cut.
  Generally, the cost of each roll depends on its type. The total cost can be written as

  $$\sum_{j=1}^{J^{\max}} \sum_{t=1} c_t^{\text{roll}} y_{tj}.$$

We note that, for each roll j, at most one term of the inner summation is non-zero (cf. constraints (5a) and (5b)).

- The costs of changing the positions of the knives.
  In general, the knife positions have to be changed if the cutting pattern used for a given roll j is different to that for the previous one. This is determined by the variables z_j and results in the cost term

$$c^{\text{change}} \sum_{j=2}^{J^{\max}} z_j,$$

  where the summation is equal to the total number of changes that are necessary.
- The cost of disposing of any trim produced.
  The width of trim produced out of raw roll j is given by the difference between the roll width and the total width of all product rolls cut from it. The former quantity depends on the type of the roll and can be expressed as $\sum_{t=1}^{T} B_t^{\text{roll}} y_{tj}$; once again, at most one of the terms in this summation can be non-zero (cf. constraints (5a) and (5b)). The latter quantity is given by $\sum_{i=1}^{I} B_i n_{ij}$. Overall, trim disposal results in the following cost term

$$c^{\text{disp}} \sum_{j=1}^{J^{\max}} \left(\sum_{t=1}^{T} B_t^{\text{roll}} y_{tj} - \sum_{i=1}^{I} B_i n_{ij} \right).$$

The above terms can now be collected in the following objective function:

$$\max \Bigg[\sum_{i=1}^{I} (p_i N_i^{\min} + \Delta_i (p_i - c_i^{\text{disc}})) - \sum_{j=1}^{J^{\max}} \sum_{t=1} c_t^{\text{roll}} y_{tj} - c^{\text{change}} \sum_{j=2}^{J^{\max}} z_j - c^{\text{disp}} \sum_{j=1}^{J^{\max}} \left(\sum_{t=1}^{T} B_t^{\text{roll}} y_{tj} - \sum_{i=1}^{I} B_i n_{ij} \right) \Bigg]. \tag{11}$$

Note that the first term in the above objective function (i.e. $\sum_{i=1}^{I} p_i N_i^{\min}$) is actually a constant and does not affect the optimal solution obtained.

3.7. Degeneracy reduction and constraint tightening

In general, the basic formulation presented above is highly degenerate: given any feasible point, one can generate many others simply by forming all possible ordering of the rolls selected to be cut. Moreover, provided all raw rolls of the same type are cut consecutively, all these feasible points will correspond to exactly the same value of the objective function.

The above property may have adverse effects on the efficiency of the search procedure. Therefore, in order to reduce the solution degeneracy without any loss of optimality, we introduce the following ordering constraints:

$$\sum_{i=1}^{I} n_{i,j-1} \geqslant \sum_{i=1}^{I} n_{ij}, \quad \forall j = 2, \ldots, J^{\max}. \tag{12}$$

This ensures that the total number of product rolls cut out of raw roll $j-1$ is never lower than the corresponding number for roll j; all completely unused raw rolls are left last in this ordering.

An alternative would be to order the raw rolls in non-increasing utilization order, i.e.

$$\sum_{i=1}^{I} B_i n_{i,j-1} \geqslant \sum_{i=1}^{I} B_i n_{ij}, \quad \forall j = 2, \ldots, J^{\max}.$$

However, our practical experience indicates that this is not as effective as constraint (12).

We also note that the constraints (7) implicitly impose a lower bound on the total number of product rolls $\sum_{i=1}^{I} n_{ij}$ cut out of a raw roll j. A stronger bound may sometimes be obtained by considering a roll of type t being used at the minimum possible engagement to produce the widest possible product rolls. This leads to the constraints

$$\sum_{t=1}^{T} \left\lceil \frac{B_t^{\min}}{\max_i B_i} \right\rceil y_{tj} \leqslant \sum_{i=1}^{I} n_{ij}, \quad \forall j = 1, \ldots, J^{\max}. \tag{13}$$

3.8. Comments

The objective function and all constraints introduced in this section are linear. Since all variables are integer valued, the formulation presented corresponds to an integer linear programming (ILP) problem. However, constraints (9) ensure that the variables Δ_i will automatically assume integer values provided variables n_{ij} do so. Therefore, Δ_i may be treated as continuous quantities, which leaves us with a mixed integer linear programming (MILP) problem. In principle, the latter can be solved using standard MILP solvers.

In the special (but quite common) case where only one type of roll is available, constraints (5a) simply imply that y_{tj} can be fixed to 1 for all $j = 1, \ldots, J^{\min}$. Both (5a) and (5b) are otherwise redundant and may be dropped. Furthermore, the lower bound on constraint (7) is directly included in the bounds of the corresponding slack variable, $[0; B_t^{\max} - B_t^{\min}]$, which results in one less constraint for each roll j.

4. Example problems

In this section, we consider four example problems of increasing complexity in order to investigate the computational behavior of our formulation. Furthermore an industrial case study is also presented. In all cases, we assume that the maximum raw roll engagement $B_t^{\max}$ is equal to the corresponding roll width B_t^{roll}. The GAMS/CPLEX vs 6.0 solver has been used for the solution [15] and all computations were carried out on a AlphaServer 4100. An integrality gap of 0.1% was assumed for the solution of all problems.

4.1. Example 1

Our first example is based on that given by Harjunkoski [9]. Some translation of the various cost coefficients was necessary to account for slight differences in the objective functions used by the two formulations. Also note that the objective used by those authors is the minimization of cost as opposed to the maximization of profit; therefore, the sign of their objective function is opposite to that of ours.

Table 1
Raw roll characteristics for Example 1

Roll type	Width B_t^{roll}	Max. spill $B_t^{max} - B_t^{min}$	Max. cuts N_t^{max}	Cost c_t^{roll}
1	1900 mm	200 mm	5	£1900

Table 2
Production data for Example 1

Product roll type i	Width B_i (mm)	Min. quantity N_i^{min}	Max. quantity N_i^{max}	Price p_i	Discount c_i^{disc}
1	330	8	10	£297.0	£0
2	360	7	8	£324.0	£0
3	385	12	13	£346.5	£0
4	415	11	11	£373.5	£0

The problem aims to determine the optimal cutting pattern for producing four different types of product rolls from a single type of raw roll. The characteristics of the latter are shown in Table 1. The production requirements are summarized in Table 2. The cost for changing the cutting pattern is £1, while disposing of the trim incurs no cost.

Harjunkoski [9] assumed a maximum of four different cutting patterns, which results in a reduction of the size of the model. In our case, the number of such patterns is determined by the solution. Also from expressions (1) and (2), we determine $J^{max} = 10$ and $J^{min} = 8$. We therefore fix $y_{1j} = 1$, for $j = 1, \ldots, 8$.

The solution we obtain is the same as that reported by Harjunkoski [9] involving the production of the minimum ordered amounts of product rolls plus one extra roll of type 2 and another of type 3. The solution is presented pictorially in Fig. 1[1] and corresponds to an objective function value of − £1622.0; thus, with the given economic data the operation incurs a loss.

The optimal solution (within a margin of optimality of 0.1%) is found within less than 1 CPU s at node 49 of the branch-and-bound algorithm using a breadth first search strategy. It must be noted that the integrality gap of our formulation is comparable to that for one of the formulations presented by Harjunkoski [9] despite the fact that it does not employ any a priori enumeration of the cutting patterns. Our formulation also examines a small number of nodes in order to detect the optimal point (Table 3).

[1] Each vertical bar corresponds to a different roll being cut. The corresponding roll type is shown immediately above each bar. Each bar is divided into a number of segments corresponding to sheets of the indicated type. The dark shaded segment at the top of the bar is the trim-loss, the percentage width of which is indicated numerically at the bottom of the figure.

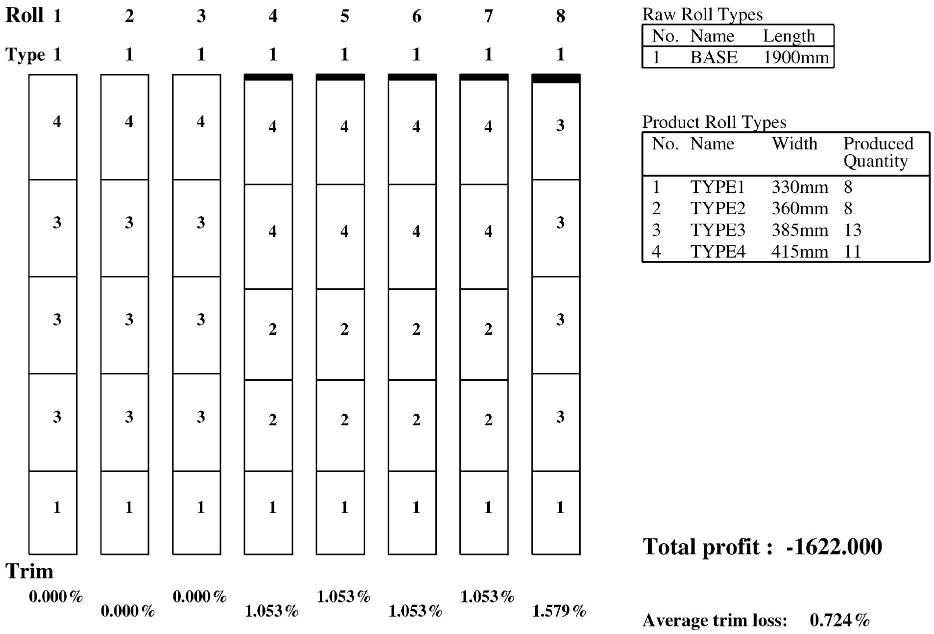


Fig. 1. Solution of Example 1.

Table 3
Computational statistics for Example 1

Optimal objective	Fully relaxed objective	Integrality gap (%)	Nodes in B&B	Variables (Bin/Int/Cont)	Constraints
(−)1622.0	(−)1619.4	0.18	49	60(13/44/3)	115

4.2. Example 2

The data for this problem are given in Tables 4 and 5. There is no cost for changing the cutting pattern or for disposing of trim. Using expressions (1) and (2), it is possible to calculate a priori that the number of raw rolls required will be between 11 and 15.

Although this problem involves only 9 types of product rolls, there are a total of 3971 different cutting patterns, all of which are feasible with respect to the minimum and maximum allowed total engagement of the rolls, the maximum number of knives that can be applied to a roll and the maximum quantities of sheets ordered. Thus, any formulation that relies on explicit pattern enumeration would have to involve a large number of discrete variables. This is, of course, a well-known problem with the classical approach to the cutting stock problem.

Our algorithm obtains the exact (0% optimality margin) optimal solution for this problem within less than 1 CPU s. This solution is presented in Fig. 2. Computational performance statistics are given in Table 6.

Table 4
Raw roll characteristics for Example 2

Roll type	Width B_t^{max}	Max. spill $B_t^{max} - B_t^{min}$	Max. cuts N_t^{max}	Cost c_t^{roll}
1	1900 mm	200 mm	5	£1600

Table 5
Production data for Examples 2 and 3

Product roll type i	Width B_i (mm)	Min. quantity N_i^{min}	Max. quantity N_i^{max}	Price p_i	Discount c_i^{disc}
1	340	8	10	£ 340	£ 0
2	365	7	8	£ 365	£ 0
3	385	12	13	£ 385	£ 0
4	415	1	11	£ 415	£ 0
5	435	5	5	£ 435	£ 0
6	260	6	8	£ 260	£ 0
7	300	4	4	£ 300	£ 0
8	320	7	8	£ 320	£ 0
9	335	3	3	£ 335	£ 0

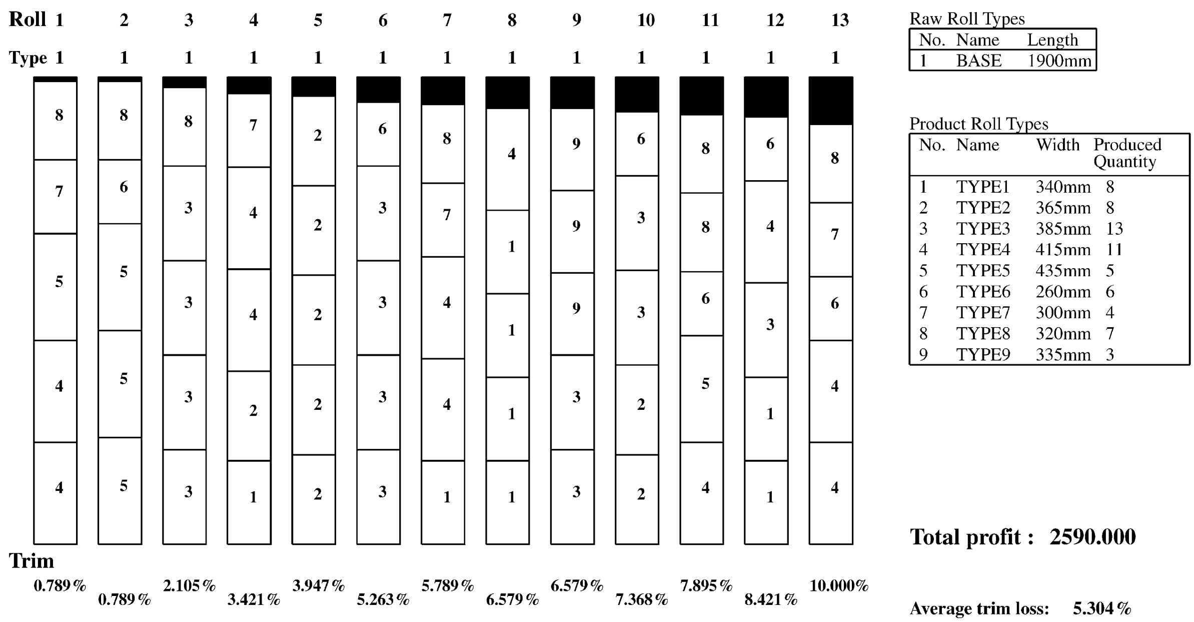


Fig. 2. Solution of Example 2.

Table 6
Computational statistics for Examples 2, 3 and 4

Formulation	Optimal objective	Fully relaxed objective	Integrality gap (%)	Nodes in B&B	Variables (Int/Bin/Cont)	Constraints
Example 2	2590.0	2630.0	1.54	43	173(6/162/5)	61
Example 3	3030.0	3030.0	0	131	203(36/162/5)	116
Example 4	1240.0	1279.3	3.16	14,800	173(6/162/5)	61

Table 7
Raw roll characteristics for Example 3

Roll type	Width $B_t^{\max}$	Max. spill $B_t^{\max} - B_t^{\min}$	Max. cuts $N_t^{\max}$	Cost c_t^{roll}
1	1900 mm	200 mm	5	£ 1600
2	2200 mm	250 mm	6	£ 1850

4.3. Example 3

This example is similar to Example 2, the only difference being that up to 6 raw rolls of a different type are now also available to be cut (see Table 7). Since a wider roll is now available, the minimum number $J^{\min}$ of required rolls is reduced to 9 (from 11 in Example 2), but the maximum number $J^{\max}$ of rolls remains the same, namely 15.

The exact (0% optimality margin) optimal solution was obtained in less than 1 CPU s. The solution is presented in Fig. 3 with the computational performance statistics shown in Table 6. The maximum possible number of raw rolls of type 2 is used. It is interesting to mention that, if no upper limit on the number of raw rolls of type 2 is imposed, only rolls of this type are actually engaged. This then results in a higher profit of £ 3380.

4.4. Example 4

This example is similar to Example 2, the only difference being that the cost coefficients for changing the cutting pattern, c^{change} and for disposing of waste trim, c^{disp} are equal to 10 and 1, respectively. The computational performance is shown in Table 6. We observe that a considerable larger number of nodes in the branch-and-bound tree is required comparing with Example 2. However, the integrality gap is relatively small.

4.5. Example 5

One justified concern with our formulation is the way in which the computational cost may increase with the number of orders that have to be satisfied. This is because more orders will

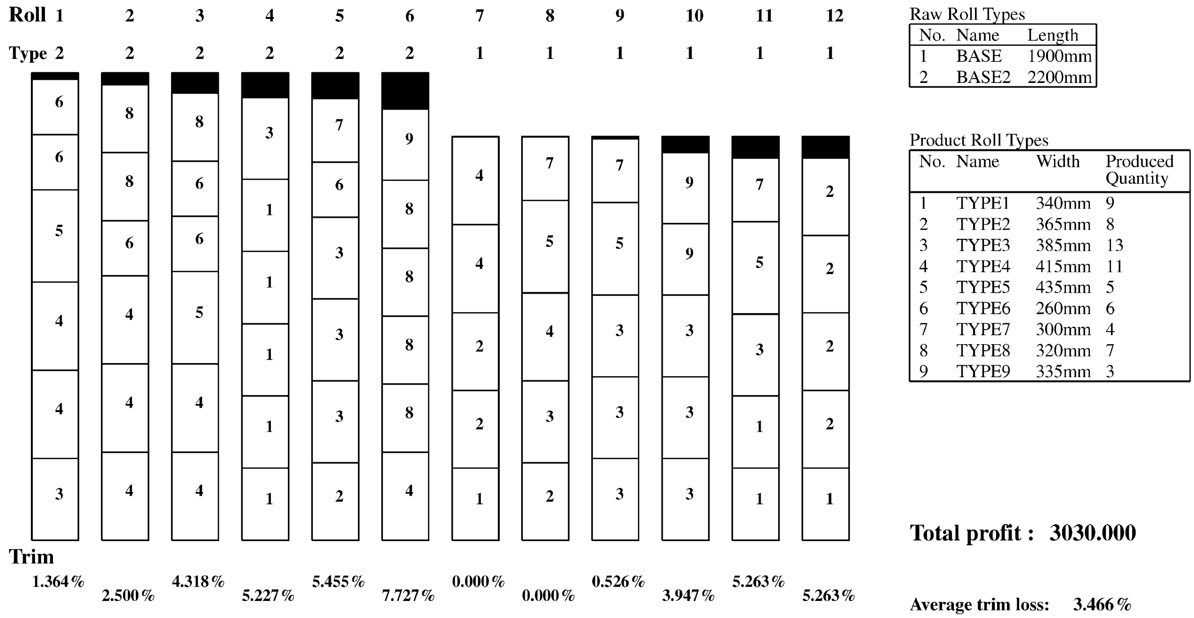


Fig. 3. Solution of Example 3.

Table 8
Computational statistics for Example 5

Example 2 orders	Optimal objective	Fully relaxed objective	Nodes in B&B	CPU (s)	Variables int/cont	Constraints	Number of rolls		
							Min.	Max.	Used
Original	2590	2630	43	< 1	178/5	64	11	15	13
Twice	5260	5260	206	< 1	336/5	115	21	30	27
4-times	10,520	10,520	828	3	623/5	208	43	59	54
10-times	26,300	26,300	5610	20	1500/5	493	106	146	134
20-times	52,600	52,600	6250	31	3519/5	893	248	293	268

generally imply more raw rolls having to be considered for cutting, (i.e., higher $J^{\max}$). The number of variables and constraints in our formulation increases linearly with the latter.

In order to study how the computational performance of the presented formulation varies with the number of ordered product rolls, we carried out three additional experiments using the original data of Example 2 but multiplying the ordered quantities (cf. Table 8) by factors of 2, 4, 10 and 15, respectively.

The results are summarized in Table 8. As expected the bigger the number of ordered product rolls, the larger the resulting mathematical problem and the difficulty of its solution. For instance, the optimal solution of the problem with twice the number of orders makes use of 25 different cutting patterns which are automatically determined by the algorithm. However, even with the largest problem (involving the production of 1340 product sheets from 268 raw rolls), it is still possible to determine the optimal solution in less than 1 min of computation on a desktop workstation. The integrality gap also remains very small in all examples. The same problem was also solved for the case where there are two different types of rolls available (as in Example 3) and the total number of orders exceeds 600 sheets. The solution corresponds to an objective function of £ 31550 with zero integrality gap. The total number of sheets produced is 660 from 120 raw rolls. The problem involves 1823 integer variables and the solution was obtained in 43 CPU s.

4.6. Industrial case study

This is an industrial case study based on a daily trim-loss optimization problem at Macedonian Paper Mills (MEL) S.A. in Northern Greece. MEL is one of the major paper-producing companies in Greece with an annual production of more than 100,000 tons. A daily order typically includes 5–15 different types of product rolls with a total weight of 10–100 tons. So far, minimization of trim-loss has been performed using heuristic-based techniques and human expertise with an average trim-loss of 4–7% depending on the order. The data for this problem are given in Tables 9 and 10 and correspond to approximated values.

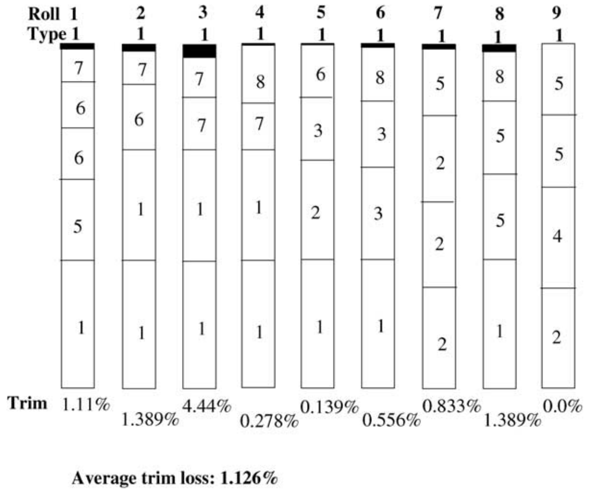
Assuming no cost for disposing of waste trim and for changing the cutting pattern the solution is depicted Fig. 4. Note that 9 raw rolls are required to satisfy the production. A total number of 1100 nodes were examined in the branch-and-bound tree requiring a computational time of less

Table 9
Raw roll characteristics for industrial case study

Roll type	Width $B_t^{\max}$	Max. spill $B_t^{\max} - B_t^{\min}$	Max. cuts $N_t^{\max}$	Cost c_t^{roll}
1	360 cm	40 cm	9	£ 515

Table 10
Production data for the industrial case study

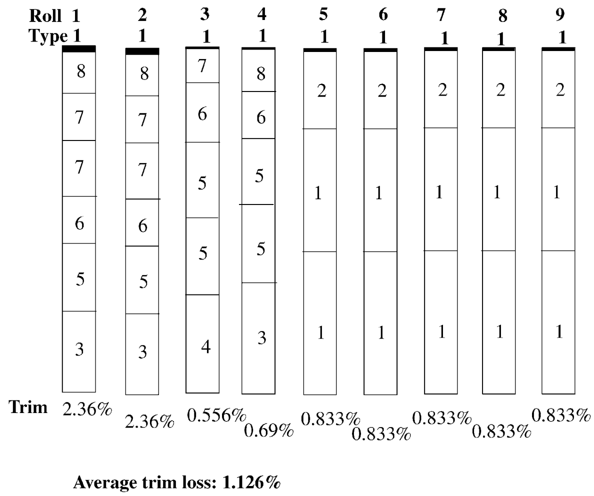
Product roll type i	Width B_i (cm)	Min. quantity $N_i^{\min}$	Max. quantity $N_i^{\max}$	Price p_i	Discount c_i^{disc}
1	133	10	10	£ 309	£ 0
2	91	5	5	£ 211	£ 0
3	85.5	3	3	£ 177	£ 0
4	101	1	1	£ 209	£ 0
5	84	6	6	£ 240	£ 0
6	50	4	4	£ 143	£ 0
7	39	5	5	£ 93	£ 0
8	54	3	3	£ 128	£ 0



Product Roll Types

No	Name	Width	Produced Quantity
1	TYPE1	133cm	10
2	TYPE2	91cm	5
3	TYPE3	85.5cm	3
4	TYPE4	101cm	1
5	TYPE5	84cm	6
6	TYPE6	50cm	4
7	TYPE7	39cm	5
8	TYPE8	54cm	3

Fig. 4. Solution of industrial case study without waste disposal and cutting change cost.



Product Roll Types

No	Name	Width	Produced Quantity
1	TYPE1	133cm	10
2	TYPE2	91cm	5
3	TYPE3	85.5cm	3
4	TYPE4	101cm	1
5	TYPE5	84cm	6
6	TYPE6	50cm	4
7	TYPE7	39cm	5
8	TYPE8	54cm	3

Fig. 5. Solution of industrial case study with waste disposal and cutting change cost.

than 5 CPU s. The optimal value of the objective function which represent the profit is £3105.8 and it is equal to the fully relaxed objective. The average trim-loss is 1.126% and represents approximately a 55% improvement comparing with the current practice based on purely human expertise.

The same problem was also solved by assuming that the cost coefficients for disposing of waste and for changing the cutting patterns are equal to 0.39 and 58.8, respectively (see Fig. 5). The optimal value of the profit is now £2842 while the value of the fully relaxed problem is £3044. Approximately 9000 nodes were considered requiring a computational time of 1 CPU min. It is interesting to note that, since the cost for changing the cutting pattern is taken into account, only three such changes take place while the average trim-loss remains the same with the previous case. However, it should be emphasized that the time savings in cutting, due to simultaneous minimization of changing the cutting pattern, has significant impact on the profitability of the plant. This is because the production rate increases almost proportionally with the reduction of total cutting time.

5. Conclusions

This paper has presented a new mathematical formulation for the determination of optimal cutting patterns in one-dimensional problems. Its main advantage over earlier formulations lies in its small integrality gap and its fewer variables and constraints. This is achieved without the need to resort to a priori generated cutting patterns, and the combinatorial increase in problem size arising from such an approach.

The formulation results in MILP problems of modest size that are within the scope of currently available commercial solvers. The integrality gap of the formulation is generally small, although the difficulty of solution increases with problems involving changeover and waste disposal costs.

Both the formulation presented here and most of the earlier ones reviewed in this paper are primarily concerned with ensuring that the various orders are fulfilled. Only the work of Westerlund [13] actually considers the *times* at which such orders are due. One interesting aspect of the presented formulation is that it explicitly characterizes the sequence of rolls that have to be cut in terms of both the type of each raw roll and the cutting pattern used for it. This opens the possibility for introducing additional variables and constraints characterizing the time at which each roll is to be cut, thereby determining the optimal cutting schedule.

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