摘　要

本文讨论了常微分算子的辛几何刻划与加权的 **Poincaré** 不等式，主要内容是:

1．考虑二阶实系数常微分算子 $L(y)=-(p(x)y')'+q(x)y(x\in I)$. 利用辛几何，对 $l(y)$ 的自伴域进行了分类，给出了 $l(y)$ 自伴域是 k- 级的充要条件.

2．考虑高阶常型实系数微分算子 $L(y)=\sum_{k=0}^{n}(p_{n-k}y^{(k)})^{(k)}(x\in[a,b])$. 利用辛几何，对 $l(y)$ 的自伴域进行了分类，给出了 $l(y)$ 自伴域是 k- 级的充要条件 $(0\le k\le n)$.

3．讨论了 J －对称常微分算子的 J －对称扩张的 J －辛几何刻划.

4．在加权 Sobolev 空间 $W^{m,p}(\Omega;w,v_{\{\alpha\}})$ 中讨论了加权的 Poincaré 不等式，给出了加权的 Poincaré 不等式成立的充分与必要条件.

5．在一维无界域上讨论了加权的 Poincaré 不等式.

6．利用一阶 Melnikov 函数，讨论了扰动系统的 Hopf 环性数.

关键词 微分算子；自伴域；辛几何；子流形； Poincaré 不等式； Hopf 分支；环性数

Abstract

The purpose of this report is to research some on complex symplectic geometry characterization of ordinery differential operators and weighted Poincaré inequalities.

1.Let $l(y) = -(py')' + qy$ be a real symmetric differential expression defined on interval I. In $L^2(I)$, we classify the self-adjoint domains generated by $l(y)$ and give the complete characterization for k–grade self-adjoint domains with complex symplectic geomety.

2.Let $L(y) = \sum_{k=0}^{n}(p_{n-k}y^{(k)})^{(k)}$ be a real symmetric differential expression defined on interval $I = [a, b]$. In $L^2(I)$, we classify the self-adjoint domains generated by $l(y)$ and give the complete characterization for self-adjoint domains with complex symplectic geomety.

3.We give complex J-symplectic geometry characterizations for J-symmetric extensions of J-symmetric ordinary differential operators.

4.we discuss the weighted Poincaré inequalities in weighted Sobolev spaces $W^{m,p}(\Omega; w, v_\alpha)$ and give some necessary and sufficient conditions for them to hold.

5.we discuss the weighted Poincaré inequalities on one-dimensional unbounded domains and give sufficient conditions for them to hold.

6.we discuss the maximal number of limit cycles which appear under perturbations in Hopf bifurcations by using degenerate first-order Melnikov function with multiple parameters.

Keywords Differential operator; Self-adjoint domain; Symplectic geomety; Submanifold;Poincaré inequalities; Hopf bifurcation,cyclicity

二阶常微分算子自伴域的辛几何刻划

摘　要　考虑二阶实系数常微分算子 $L(y)=-(p(x)y')'+q(x)y$ $(x\in I)$. 利用辛几何，对 $l(y)$ 的自伴域进行了分类，给出了 $l(y)$ 自伴域是 k- 级的充要条件.

关键词　微分算子；自伴域；辛几何；子流形
MR(1991) 主题分类　34B05, 34L05, 47B05, 58F05
中图分类　O175.3

1 引言

设
$$l(y)=-(py')'+qy$$
是 I 上的二阶实系数微分算式， p,p',q 在 I 上连续，且 $p(x)>0$.

当 $I=[a,b]$ 时，由 [11] 知， $l(y)$ 的亏指数必为 (2,2) 且必可生成自伴算子. 如何去描述 $l(y)$ 的自伴域呢？ 1954 年， E.A.Coddington 在 [13] 中给出了这个问题的完全解答. 同一时期， M.A. Naimark 在 [4] 中给出了由“拟导数”定义的对称微分算子自伴域的完全刻划. 1962, W.N.Everitt 在 [14] 中应用微分方程 $l(y)=\lambda y$ 的解给出了自伴域的描述.

当 $I=[a,+\infty)$ 时，由 H.Weyl 和 E.C.Titchmarsh 关于二阶奇型自伴微分算子的经典理论 [13][14] 知， $l(y)$ 的亏指数仅为 (1,1) 与 (2,2) 两种情形，前者称为极限点型，后者称为极限园型. 当 $l(y)$ 为极限点型时， Weyl-Titchmarsh 域是 $l(y)$ 自伴域的完全描述. 当 $l(y)$ 为极限园型时， Weyl-Titchmarsh 域仅是 $l(y)$ 自伴域的一种特殊描述. 如何得到 $l(y)$ 自伴域的完全描述呢？ 1982 年，曹之江教授在 [2] 中，给出了 $l(y)$ 自伴域直接而完全的描述 (以下简称为 Cao 域), 并证明了 Weyl-Titchmarsh 域作为一种特例包含在 Cao 域中，从而完全解决了二阶奇型自伴微分算子的解析描述问题. 1999 年， W.N.Everitt 和 L.Markus 在文 [1] 中利用辛几何也给出了微分算子 $l(y)$ 自伴扩张的完全刻划.

微分算子理论是当代量子力学的数学支柱，是解决数学物理方程以及大量科学技术应用问题的重要数学工具. 微分算子自伴扩张问题是微分

算子理论的基础问题之一，受到大家的广泛关注，如文 [1–14] 等. 以前的大部分研究工作是利用分析、算子等方法对自伴域进行描述，并未对自伴域进行分类.

本文讨论二阶实系数对称常微分算子 $l(y)$，利用辛几何的方法，对其自伴域进行了分类，同时也给出了自伴域是 k- 级的充分必要条件.

2 预备知识

定义 2.1 一个复的辛空间 $\mathbf{S}$ 是一个复的线性空间，且带有一个辛形式 $[:]$，即

(1) $[:]$ 是一个半双线性型，

$$u, v \to [u:v], \mathbf{S}\times\mathbf{S} \to C, [c_1 u + c_2 v : w] = c_1[u:w] + c_2[v:w],$$

(2) $[:]$ 是一个反 Hermitian 型，

$$[u:v] = -\overline{[v:u]}, [u: c_1 v + c_2 w] = \overline{c_1}[u:v] + \overline{c_2}[u:w],$$

(3) $[:]$ 是非退化的，

$$[u:\mathbf{S}] = 0 \Longrightarrow u = 0,$$

对 $\forall u_1 v_1 w \in \mathbf{S}, \forall c_1, c_2 \in C$.

定义 2.2 复辛空间 $\mathbf{S}$ 的一个线性流形 $\mathbf{L}$ 被称为是 Lagrangian 的，若 $[\mathbf{L}:\mathbf{L}] = 0$，即，对 $\forall u, v \in \mathbf{L}$ 有 $[u:v] = 0$.

$\mathbf{S}$ 的一个 Lagrangian 子流形 $\mathbf{L}$ 被称为是完全的，若 $u \in \mathbf{S}$ 且 $[u:\mathbf{L}] = 0 \Longrightarrow u \in \mathbf{L}$.

定义 2.3 设 $\mathbf{S}$ 是一个复辛空间. 若 $\mathbf{S}_-$ 和 $\mathbf{S}_+$ 是 $\mathbf{S}$ 的线性子流形，且满足：

(1) $\mathbf{S} = \text{span}\{\mathbf{S}_-, \mathbf{S}_+\}$;

(2) $[\mathbf{S}_- : \mathbf{S}_+] = 0$;

则称 $\mathbf{S}_-$ 和 $\mathbf{S}_+$ 在 $\mathbf{S}$ 中是辛正交互补，记作 $\mathbf{S} = \mathbf{S}_- \oplus \mathbf{S}_+$.

有关辛几何的概念详细见文 [1].

3 二阶常型微分算子

设 $I = [a, b]$，由 $l(y)$ 生成的最大算子与最小算子定义如下：

$T_{\max}(y) = l(y), y \in D(T_{\max}) = \{y : I \to C | y, y' \in AC(I)$ 且 $l(y) \in L^2(I)\}$;

$T_{\min}(y) = l(y), y \in D(T_{\min}) = \{y \in D(T_{\max}) | y(a) = y'(a) = y(b) = y'(b) = 0\}$.
由常微分算子理论知，$T_{\min}$ 和 $T_{\max}$ 是闭线性算子，且 $T_{\max}^* = T_{\min}, T_{\min}^* = T_{\max}^*$.

令

$$\mathbf{S} = D(T_{\max})/D(T_{\min}),$$

在 $\mathbf{S}$ 中定义辛形式 [:] 为：

$$[\mathbf{f} : \mathbf{g}] = [f + D(T_{\min}) : g + D(T_{\min})] = [fg]_a^b, \quad \forall f, g \in D(T_{\max})$$

其中 $\mathbf{f} = f + D(T_{\min}), \mathbf{g} = g + D(T_{\min}) \in \mathbf{S}$, $[fg]_a^b$ 是 f 与 g 的契合式. 若令

$$[fg]_a^b = [f : g],$$

则由文 [1] 和 [11] 知， $D(T_{\min})$ 可表示为

$$D(T_{\min}) = \{f \in D(T_{\max}) | [f : D(T_{\max})] = 0\},$$

$T_{\min}$ 是一个对称算子.

由文 [1] 可得如下几个引理：

引理 3.1 $\mathbf{S} = D(T_{\max})/D(T_{\min})$ 是一个复辛空间，且 $\dim \mathbf{S} = 4$.

引理 3.2 $\mathbf{S} = \mathbf{S}_- \oplus \mathbf{S}_+$, 其中

$$\mathbf{S}_- = \{\mathbf{f} \in \mathbf{S} | f(b) = f'(b) = 0\},$$
$$\mathbf{S}_+ = \{\mathbf{f} \in \mathbf{S} | f(a) = f'(a) = 0\},$$

且 $\dim \mathbf{S}_- = \dim \mathbf{S}_+ = 2$.

引理 3.3 (平衡相交原理) 若 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形，则

$$\begin{aligned} 0 &\le \frac{1}{2}\dim \mathbf{S}_- - \dim \mathbf{L} \cap \mathbf{S}_- = \frac{1}{2}\dim \mathbf{S}_+ - \dim \mathbf{L} \cap \mathbf{S}_+ \\ &\le \frac{1}{2}\min\{\dim \mathbf{S}_-, \dim \mathbf{S}_+\}, \end{aligned}$$

即

$$0 \le 1 - \dim \mathbf{L} \cap \mathbf{S}_- = 1 - \dim \mathbf{L} \cap \mathbf{S}_+ \le 1.$$

定义 3.1 设 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形，令

$$k = 1 - \dim \mathbf{L} \cap \mathbf{S}_- = 1 - \dim \mathbf{L} \cap \mathbf{S}_+.$$

则称 $\mathbf{L}$ 是 k- 级的，也称 $D(T_{\mathbf{L}})$ 是 k- 级的.

引理 3.4 (GKN 定理) (1) $T_{\min}$ 有自伴扩张 $\Longleftrightarrow$ $\mathbf{S}$ 有完全 Lagrangian 子流形.

(2) $\mathbf{S}$ 的 Lagrangian 子流形 $\mathbf{L}$ 是完全的 $\Longleftrightarrow \dim \mathbf{L} = \frac{1}{2}\dim \mathbf{S} = 2$.

(3) 若 T 是 $T_{\min}$ 的一个自伴扩张，自伴域为 $D(T)$, 则 S 有唯一的完全 Lagrangian 子流形 $\mathbf{L_T}$ 与其对应，使得

$$\mathbf{L_T} = D(T)/D(T_{\min}).$$

(4) 若 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形，则 $T_{\min}$ 有唯一的自伴扩张 $T_{\mathbf{L}}$ 与其对应，使得

$$D(T_{\mathbf{L}}) = c_1 f_1 + c_2 f_2 + D(T_{\min}),$$

其中 $\mathbf{f_1}, \mathbf{f_2}$ 是 $\mathbf{L}$ 的一个基， $f_1, f_2 \in D(T_{\max})$,c_1, c_2 是任意复数.

注 由引理 3.4 知，讨论 $T_{\min}$ 的自伴扩张问题等价于讨论复辛空间 $\mathbf{S} = D(T_{\max})/D(T_{\min})$ 的完全 Lagrangian 子流形. 因此，对 S 的完全 Lagrangian 子流形进行的分类和描述就等价于对 $l(y)$ 的自伴域进行的分类和描述.

定理 3.1 $\mathbf{S}$ 的完全 Lagrangian 子流形有且仅有 0- 级与 1- 级的.

证明 由引理 3.3 与文 [1] 的定理 5 知.

因为 $\dim \mathbf{S} = 4$, 所以 $\mathbf{S}$ 与 C^4 线性同构. 因此我们可以利用 C^4 的单位基向量

$$e^1 = (1,0,0,0),\ e^2 = (0,1,0,0),\ e^3 = (0,0,1,0),\ e^4 = (0,0,0,1)$$

来表示 $\mathbf{S}$, 即 $\mathbf{S} = \mathrm{span}\{e^1, e^2, e^3, e^4\}$. 设 $\mathbf{f} \in \mathbf{S}$, 则可以如下选取 $\mathbf{f}$ 的坐标:

$$\mathbf{f} = (f(a), p(a)f'(a), f(b), p(b)f'(b)) = f(a)e^1 + p(a)f'(a)e^2 + f(b)e^3 + p(b)f'(b)e^4,$$

使得有结论:

定理 3.2 对于 $\forall \mathbf{f}, \mathbf{g} \in \mathbf{S}$, 有

$$[\mathbf{f} : \mathbf{g}] = \mathbf{f} H \mathbf{g}^*,$$

其中

$$H=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}.$$

证明 由 $\mathbf{S}$ 的辛形式的含义有：

$$[\mathbf{f}:\mathbf{g}]=[fg]_a^b=\langle l(f),g\rangle-\langle f,l(g)\rangle=(-pf'\bar{g}+pf\bar{g}')|_a^b=\mathbf{f}H\mathbf{g}^*.$$

定理 3.3 $\mathbf{S}_-=\mathrm{span}\{e^1,e^2\},\mathbf{S}_+=\mathrm{span}\{e^3,e^4\}$.

证明 下面证明 $\mathbf{S}_-=\mathrm{span}\{e^1,e^2\}$.

对于 $\forall\mathbf{f}\in\mathbf{S}_-$, 则 $\mathbf{f}\in\mathbf{S}, f\in D(T_{\max})$ 且 $f(b)=f^1(b)=0$. 因此

$$\mathbf{f}=f(a)e^1+p(a)f^1(a)e^2+f(b)e^3+p(b)f^1(b)e^4=f(a)e^1+p(a)f^1(a)e^2,$$

$\mathbf{f}\in\mathrm{span}\{e^1,e^2\}$. 故 $\mathbf{S}_-\subseteq\mathrm{span}\{e^1,e^2\}$. .

若 $\mathbf{f}\in\mathrm{span}\{e^1,e^2\}$, 则 $\mathbf{f}=f(a)e^1+p(a)f^1(a)e^2+0e^3+0e^4$, 即 $f(b)=f^1(b)=0$. 因此 $\mathrm{span}\{e^1,e^2\}\subseteq\mathbf{S}_-$. 于是 $\mathbf{S}_-=\mathrm{span}\{e^1,e^2\}$.

类似可证 $\mathbf{S}_+=\mathrm{span}\{e^3,e^4\}$.

定理 3.4 $\mathbf{L}$ 是 $\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形 $\Longleftrightarrow\exists a_1,a_2,b_1,b_2\in C$, 使得 $\mathbf{L}=\mathrm{span}\{a_1e^1+a_2e^2,b_1e^3+b_2e^4\}$, 且满足

(1) a_1 与 a_2 不全为 0, b_1 与 b_2 不全为 0;

(2) $\begin{vmatrix}a_1&a_2\\\bar{a}_1&\bar{a}_2\end{vmatrix}=\begin{vmatrix}b_1&b_2\\\bar{b}_1&\bar{b}_2\end{vmatrix}=0$.

证明($\Longleftarrow$) 对于任意 $\mathbf{f},\mathbf{g}\in\mathbf{L}$, 则存在 $\alpha_1,\alpha_2,\beta_1,\beta_2\in C$, 使得

$$\begin{aligned}\mathbf{f}=&\alpha_1(a_1e^1+a_2e^2)+\beta_1(b_1e^3+b_2e^4)=\alpha_1a_1e^1+\alpha_1a_2e^2+\beta_1b_1e^3+\beta_1b_2e^4\\\mathbf{g}=&\alpha_2(a_1e^1+a_2e^2)+\beta_2(b_1e^3+b_2e^4)=\alpha_2a_1e^1+\alpha_2a_2e^2+\beta_2b_1e^3+\beta_2b_2e^4,\end{aligned}$$

则由定理 3.2 和条件 (2) 得

$$[\mathbf{f}:\mathbf{g}]=(\alpha_1a_1,\alpha_1a_2,\beta_1b_1,\beta_1b_2)H(\alpha_2a_1,\alpha_2a_2,\beta_2b_1,\beta_2b_2)^*=0$$

故 $[\mathbf{L}:\mathbf{L}]=0$, 即 $\mathbf{L}$ 是 $\mathbf{S}$ 的 *Lagrangian* 子流形. 由 (1) 知, $\dim\mathbf{L}=2$. 因此, 由引理 3.4 知, $\mathbf{L}$ 是 $\mathbf{S}$ 的完全 Lagrangian 子流形. 由 (1) 知,

$$\dim\mathbf{L}\cap\mathbf{S}_-=\dim\mathbf{L}\cap\mathbf{S}_+=1,$$

所以 $\mathbf{L}$ 是 0- 级的.

($\Longleftarrow$) 因为 $\mathbf{L}$ 是 $\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形，所以

$$\dim \mathbf{L} = 2, \dim \mathbf{L} \cap \mathbf{S}_- = \dim \mathbf{L} \cap \mathbf{S}_+ = 1, [\mathbf{L} : \mathbf{L}] = 0.$$

因此 $\exists a_1, a_2, b_1, b_2 \in C$，使得

$$\mathbf{L} = \operatorname{span}\{a_1 e^1 + a_2 e^2, b_1 e^3 + b_2 e^4\}$$

易证明 $\mathbf{L}$ 满足条件 (1) 与 (2).

推论 3.5 $\mathbf{L} = \operatorname{span}\{a_1 e^1 + a_2 e^2, b_1 e^3 + b_2 e^4\}$ 是 $\mathbf{S}$ 的 0- 级完全 Langrangian 子流形 $\Longleftrightarrow \exists a_1, a_2, b_1, b_2 \in C$，使得 $\mathbf{L}$ 满足以下条件之一：

(1) 当 a_1 与 a_2 中仅有一个数为零，b_1 与 b_2 中仅有一个数为零时，不妨设 $a_2 = b_2 = 0$，则

$$\mathbf{L} = \{\mathbf{f} \in \mathbf{S} | f'(a) = f'(b) = 0\}.$$

(2) 当 a_1, a_2, b_1, b_2 中只有一个数为零时，不妨设 $b_2 = 0$，则

$$\mathbf{L} = \{\mathbf{f} \in \mathbf{S} | a_2 f(a) = a_1 p(a) f'(a), f'(b) = 0\},$$

且 $\begin{vmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = 0.$

(3) 当 a_1, a_2, b_1, b_2 全非零时，则

$$\mathbf{L} = \{\mathbf{f} \in \mathbf{S} | a_2 f(a) = a_1 p(a) f'(a), b_2 f(b) = b_1 p(b) f'(b)\},$$

且 $\begin{vmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = 0.$

注 从推论 3.5 可以看出，$\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形可以用边界条件来描述，但边界条件全部是分离的.

例 3.1

$$\mathbf{L} = \operatorname{span}\{e^1 + e^2, e^3 + e^4\} = \{\mathbf{f} \in \mathbf{S} | f(a) = p(a) f'(a), f(b) = p(b) f'(b)\}$$

是 $\mathbf{S}$ 的一个 0- 级完全 Lagrangian 子流形.

定理 3.6 $\mathbf{L}$ 是 $\mathbf{S}$ 的 1- 级完全 Lagrangian 子流形 $\Longleftrightarrow \exists a_i, b_i \in C (i = 1, 2, 3, 4)$，使得

$$\mathbf{L} = \operatorname{span}\{a_1 e^1 + a_2 e^2 + a_3 e^3 + a_4 e^4, b_1 e^1 + b_2 e^2 + b_3 e^3 + b_4 e^4\}$$

且满足：

(1) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} \neq 0;$

(2) $\begin{vmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{a}_3 & \bar{a}_4 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} b_3 & b_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}.$

证明 ($\Longleftarrow$) 对于任意 $\mathbf{f}, \mathbf{g} \in \mathbf{L}$, 则存在 $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C$, 使得

$$\begin{aligned}
\mathbf{f} =&\alpha_1(a_1e^1 + a_2e^2 + a_3e^3 + a_4e^4) + \beta_1(b_1e^3 + b_2e^4 + b_3e^3 + b_4e^4) \\
=&(\alpha_1a_1 + \beta_1b_1)e^1 + (\alpha_1a_2 + \beta_1b_2)e^2 + (\alpha_1a_3 + \beta_1b_3)e^3 + (\alpha_1a_4 + \beta_1b_4)e^4 \\
\mathbf{g} =&\alpha_2(a_1e^1 + a_2e^2 + a_3e^3 + a_4e^4) + \beta_2(b_1e^3 + b_2e^4 + b_3e^3 + b_4e^4) \\
=&(\alpha_2a_1 + \beta_2b_1)e^1 + (\alpha_2a_2 + \beta_2b_2)e^2 + (\alpha_2a_3 + \beta_2b_3)e^3 + (\alpha_2a_4 + \beta_2b_4)e^4,
\end{aligned}$$

则由定理 3.2 和条件 (2) 得

$$\begin{aligned}
[\mathbf{f} : \mathbf{g}] =&(\alpha_1a_1 + \beta_1b_1, \alpha_1a_2 + \beta_1b_2, \alpha_1a_3 + \beta_1b_3, \alpha_1a_4 + \beta_1b_4)H \\
&(\alpha_2a_1 + \beta_2b_1, \alpha_2a_2 + \beta_2b_2, \alpha_2a_3 + \beta_2b_3, \alpha_2a_4 + \beta_2b_4)^* \\
=&0,
\end{aligned}$$

故 $[\mathbf{L} : \mathbf{L}] = 0$ ，即 $\mathbf{L}$ 是 $\mathbf{S}$ 的 *Lagrangian* 子流形. 由 (1) 知，秩 $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} = 2$, 于是 $\dim \mathbf{L} = 2$. 故由引理 3.4 知， $\mathbf{L}$ 是 $\mathbf{S}$ 的完全 Lagrangian 子流形. 由 (1) 知,

$$\dim \mathbf{L} \cap \mathbf{S}_- = \dim \mathbf{L} \cap \mathbf{S}_+ = 0.$$

因此 $\mathbf{L}$ 是 1- 级的.

($\Longrightarrow$) 因为 $\mathbf{L}$ 是 $\mathbf{S}$ 的 1- 级完全 Lagrangian 子流形，所以

$$\dim \mathbf{L} = 2, \dim \mathbf{L} \cap \mathbf{S}_- = \dim \mathbf{L} \cap \mathbf{S}_+ = 0, [\mathbf{L} : \mathbf{L}] = 0.$$

因此 $\exists a_i, b_i \in C(i = 1, 2, 3, 4)$, 使得

$$\mathbf{L} = \operatorname{span}\{a_1e^1 + a_2e^2 + a_3e^3 + a_4e^4, b_1e^1 + b_2e^2 + b_3e^3 + b_4e^4\}$$

易证明 $\mathbf{L}$ 满足条件 (1) 与 (2).

推论 3.7 $\mathbf{L}$ 是 $\mathbf{S}$ 的 1- 级完全 Langrangian 子流形 $\Longleftrightarrow \exists a_i, b_i \in C(i = 1, 2, 3, 4)$, 使得

$$\mathbf{L} = \{\mathbf{f} \in \mathbf{S} \mid \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} f(a) \\ p(a)f'(a) \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}^{-1} \begin{pmatrix} f(b) \\ p(b)f'(b) \end{pmatrix}\}$$

且满足:

(1) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} \neq 0;$

(2) $\begin{vmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{a}_3 & \bar{a}_4 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} b_3 & b_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}.$

证明 ($\Longrightarrow$) 设 $\mathbf{f} \in \mathbf{L}$, 则由定理 3.6 知, 存在唯一的 $\alpha_1, \alpha_2 \in C$, 使得

$$\begin{aligned}\mathbf{f} =& \alpha_1(a_1e^1 + a_2e^2 + a_3e^3 + a_4e^4) + \alpha_2(b_1e^1 + b_2e^2 + b_3e^3 + b_4e^4)\\ =&(a_1\alpha_1 + b_1\alpha_2)e^1 + (a_2\alpha_1 + b_2\alpha_2)e^2 + (a_3\alpha_1 + b_3\alpha_2)e^3 + (a_4\alpha_1 + b_4\alpha_2)e^4.\end{aligned}$$

由 $\mathbf{f} = (f(a), p(a)f'(a), f(b), p(b)f'(b))$ 得,

$$\begin{cases} a_1\alpha_1 + b_1\alpha_2 = f(a) \\ a_2\alpha_1 + b_2\alpha_2 = p(a)f'(a) \end{cases} \tag{3.1}$$

$$\begin{cases} a_3\alpha_1 + b_3\alpha_2 = f(b) \\ a_4\alpha_1 + b_4\alpha_2 = p(b)f'(b) \end{cases} \tag{3.2}$$

由定理 3.6(1) 知, 方程组 (3.1) 和 (3.2) 的解唯一, 因此

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} f(a) \\ p(a)f'(a) \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}^{-1} \begin{pmatrix} f(b) \\ p(b)f'(b) \end{pmatrix}.$$

故结论成立.

($\Longleftarrow$) 对于任意 $\mathbf{f} \in \mathbf{L}$, 令

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} f(a) \\ p(a)f'(a) \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}^{-1} \begin{pmatrix} f(b) \\ p(b)f'(b) \end{pmatrix},$$

则 (3.1),(3.2) 式成立. 故

$$\mathbf{L} = \mathrm{span}\{a_1e^1 + a_2e^2 + a_3e^3 + a_4e^4, b_1e^1 + b_2e^2 + b_3e^3 + b_4e^4\}.$$

由定理 3.6 知结论成立.

例 3.2

$$\mathbf{L} = \mathrm{span}\{e^1 + e^3, e^2 + e^4\} = \{\mathbf{f} \in \mathbf{S} | f(a) = f(b), p(a)f'(a) = p(b)f'(b)\}$$

是 $\mathbf{S}$ 的一个 1- 级完全 Lagrangian 子流形.

注 从推论 3.7 可以看出, $\mathbf{S}$ 的 1- 级完全 Lagrangian 子流形也可以用边界条件来描述, 但边界条件不是分离的, 而是耦合的.

4 二阶奇型微分算子

设 $I = [0, \infty)$, 由 $l(y)$ 生成的最大算子与最小算子定义如下:

$T_{\max}(y) = l(y), y \in D(T_{\max}) = \{y : I \to C | y, y'$ 在 $[0, \infty)$ 的任何紧子集上绝对连续， $y, l(y) \in L^2(I)\}$;

$T_{\min}(y) = l(y), y \in D(T_{\min}) = \{y \in D(T_{\max}) | y(0) = y'(0) = 0$ 且对于 $\forall z \in D(T_{\max})$ 有 $[yz](\infty) = 0\}$;
其中 $[yz](x)$ 是 $l(y)$ 的 Lagrange 双线性型，即

$$[yz](x) = \{y'(p_0\bar{z}) - y(p_0\bar{z}')\}(x).$$

由常微分算子理论知, $T_{\min}$ 和 $T_{\max}$ 是闭线性算子, 且 $T_{\max}^* = T_{\min}, T_{\min}^* = T_{\max}$.

令

$$\mathbf{S} = D(T_{\max})/D(T_{\min}),$$

在 $\mathbf{S}$ 中定义辛形式 $[:]$ 为:

$$[\mathbf{f} : \mathbf{g}] = [f + D(T_{\min}) : g + D(T_{\min})] = [fg]_0^\infty, \quad \forall f, g \in D(T_{\max})$$

其中 $\mathbf{f} = f + D(T_{\min}), \mathbf{g} = g + D(T_{\min}) \in \mathbf{S}$, $[fg]_0^\infty$ 是 f 与 g 的契合式. 若令

$$[fg]_0^\infty = [f : g],$$

则由文 [1]、 [2] 和 [11] 得,

$$D(T_{\min}) = \{f \in D(T_{\max}) | [f : D(T_{\max})] = 0\},$$

易知 $T_{\min}$ 是一个对称算子.

由文 [2] 得:

引理 4.1 设 ϕ, ψ 是 $l(y) = 0$ 的两个实解，且满足 $[\phi\psi](a) = 1$, 则 $[\phi\psi](\infty) = 1$, 且

$$D(T_{\min}) = \{f \in D(T_{\max}) | f(0) = f'(0) = 0, [f\phi](\infty) = [f\psi](\infty) = 0\}.$$

由文 [1] 可得如下几个引理:

引理 4.2 $\mathbf{S} = D(T_{\max})/D(T_{\min})$ 是一个复辛空间.

引理 4.3 $\mathbf{S} = \mathbf{S}_- \oplus \mathbf{S}_+$, 其中

$$\mathbf{S}_- = \{\mathbf{f} \in \mathbf{S} | [f\phi](\infty) = [f\psi](\infty) = 0\},$$
$$\mathbf{S}_+ = \{\mathbf{f} \in \mathbf{S} | f(0) = f'(0) = 0\}.$$

引理 4.4 (平衡相交原理) 若 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形，则

$$0 \leq \frac{1}{2}\dim \mathbf{S}_- - \dim L \cap \mathbf{S}_- = \frac{1}{2}\dim \mathbf{S}_+ - \dim L \cap \mathbf{S}_+ \leq \frac{1}{2}\min\{\dim \mathbf{S}_-, \dim \mathbf{S}_+\}.$$

定义 4.1 设 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形，令

$$k = \frac{1}{2}\dim \mathbf{S}_- - \dim \mathbf{L} \cap \mathbf{S}_- = \frac{1}{2}\dim \mathbf{S}_+ - \dim \mathbf{L} \cap \mathbf{S}_+.$$

则称 $\mathbf{L}$ 是 $k-$ 级的，也称 $D(T_{\mathbf{L}})$ 是 $k-$ 级的.

引理 4.5(GKN 定理) (1) $T_{\min}$ 有自伴扩张 $\Longleftrightarrow$ $\mathbf{S}$ 有完全 Lagrangian 子流形.

(2) $\mathbf{S}$ 的 Lagrangian 子流形 $\mathbf{L}$ 是完全的 $\Longleftrightarrow \dim \mathbf{L} = \frac{1}{2}\dim \mathbf{S}$.

(3) 若 T 是 $T_{\min}$ 的一个自伴扩张，自伴域为 $D(T)$，则 $\mathbf{S}$ 有唯一的完全 Lagrangian 子流形 $\mathbf{L_T}$ 与其对应，使得

$$\mathbf{L_T} = D(T)/D(T_{\min}).$$

(4) 若 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形，则 $T_{\min}$ 有唯一的自伴扩张 $T_{\mathbf{L}}$ 与其对应，使得

$$D(T_{\mathbf{L}}) = c_1 f_1 + \cdots + c_n f_n + D(T_{\min}),$$

其中 $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n$ 是 L 的一个基， $f_1, \cdots, f_n \in D(T_{\max}), c_1, c_2, \cdots, c_n$ 是任意复数.

注 由引理 4.5 知，讨论 $T_{\min}$ 的自伴扩张问题等价于讨论复辛空间 $\mathbf{S} = D(T_{\max})/D(T_{\min})$ 的完全 Lagrangian 子流形. 因此，对 $\mathbf{S}$ 的完全 Lagrangian 子流形的分类与描述，就等价于对 $l(y)$ 的自办域进行的分类与描述.

4.1 极限点型

引理 4.1.1 若 $l(y)$ 为极限点型，则

$$D(T_{\min}) = \{f \in D(T_{\max}) | f(0) = f'(0) = 0\}.$$

证明 设 $l(y)$ 为极限点型，则其亏指数为 (1,1). 由文 [11] 第五章引理 4.1 知，对于 $\forall f, g \in D(T_{\max})$ 有 $[fg](\infty) = 0$.

引理 4.1.2 $\mathbf{S}_+ = \mathbf{S}, \mathbf{S}_- = \{\mathbf{0}\}$.

引理 4.1.3 $\dim \mathbf{S} = \dim D(T_{\max})/D(T_{\min}) = 2$.

定理 4.1.4 $\mathbf{S}$ 只有 0- 级完全 Lagrangian 子流形.

证明 由定义 4.1 和引理 4.5, 4.1,2, 4.1.3 知.

因为 $\dim \mathbf{S} = 2$, 所以 $\mathbf{S}$ 与 C^2 线性同构. 因此我们可以利用 C^2 的单位基向量

$$e^1 = (1,0) \quad e^2 = (0,1)$$

来表示 $\mathbf{S}$, 即 $\mathbf{S} = \mathrm{span}\{e^1, e^2\}$. 设 $\mathbf{f} \in \mathbf{S}$, 则可以如下选取 $\mathbf{f}$ 的坐标:

$$\mathbf{f} = (f(0), p(0)f'(0)) = f(0)e^1 + p(0)f'(0)e^2,$$

使得有结论:

定理 4.1.5 对于 $\forall \mathbf{f}, \mathbf{g} \in \mathbf{S}$ 有

$$[\mathbf{f} : \mathbf{g}] = \mathbf{f} H \mathbf{g}^*,$$

其中

$$H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

证明 由 $\mathbf{S}$ 的辛形式的定义有

$$[\mathbf{f} : \mathbf{g}] = [fg]_0^\infty = \langle l(f), g \rangle - \langle f, l(g) \rangle = (-pf^1\bar{g} + pf\bar{g}^1)|_0^\infty = \mathbf{f} H \mathbf{g}^*.$$

定理 4.1.6 $\mathbf{L}$ 是 $\mathbf{S}$ 的 (0- 级) 完全 Lagrangian 子流形 $\Longleftrightarrow$ $\exists a_1, a_2 \in C$, 使得 $\mathbf{L} = \mathrm{span}\{a_1e^1 + a_2e^2\}$, 且满足

(1) a_1, a_2 不全为零;

(2) $\begin{vmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = 0.$

证明($\Longleftarrow$) 对于任意 $\mathbf{f}, \mathbf{g} \in \mathbf{L}$, 则存在 $\alpha, \beta \in C$, 使得

$$\begin{aligned} \mathbf{f} &= \alpha(a_1e^1 + a_2e^2) = \alpha a_1 e^1 + \alpha a_2 e^2 \\ \mathbf{g} &= \beta(a_1e^1 + a_2e^2) = \beta a_1 e^1 + \beta a_2 e^2, \end{aligned}$$

则由定理 4.1.5 和条件 (2) 得

$$[\mathbf{f} : \mathbf{g}] = (\alpha a_1, \alpha a_2) H (\beta a_1, \beta a_2)^* = 0$$

故 $[\mathbf{L} : \mathbf{L}] = 0$, 即 $\mathbf{L}$ 是 $\mathbf{S}$ 的 *Lagrangian* 子流形. 由 (1) 知, $\dim \mathbf{L} = 1$. 因此, 由引理 4.5 知, $\mathbf{L}$ 是 $\mathbf{S}$ 的 (0- 级) 完全 Lagrangian 子流形.

($\Longrightarrow$) 因为 $\mathbf{L}$ 是 $\mathbf{S}$ 的 (0- 级) 完全 Lagrangian 子流形, 所以

$$\dim \mathbf{L} = 1, \ \dim \mathbf{L} \cap \mathbf{S}_- = 0, \ \dim \mathbf{L} \cap \mathbf{S}_+ = 1$$

且 $[\mathbf{L}:\mathbf{L}]=0$.. 因此 $\exists a_1,a_2\in C$, 使得

$$\mathbf{L}=\text{span}\{a_1e^1+a_2e^2\}$$

易证明 $\mathbf{L}$ 满足条件 (1) 与 (2).

推论 4.1.7 $\mathbf{L}=\text{span}\{a_1e^1+a_2e^2\}$ 是 $\mathbf{S}$ 的 (0- 级) 完全 Lagrangian 子流形 $\Longleftrightarrow \exists a_1,a_2\in C$, 使得 $\mathbf{L}$ 满足以下条件之一:

(1) 当 a_1,a_2 中有一个为零时, 不妨设 $a_2=0$. 则

$$\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|f'(0)=0\}.$$

(2) 当 a_1,a_2 全非零时, 则

$$\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|a_2f(0)=a_1p(0)f'(0)\}$$

且满足 $\begin{vmatrix}a_1 & a_2\\ \bar{a}_1 & \bar{a}_2\end{vmatrix}=0$.

4.2 极限园型

设 $l(y)$ 为极限园型, 则其亏指数为 (2,2). 由文 [1] 可知

引理 4.2.1 $\dim\mathbf{S}=4, \dim\mathbf{S}_-=\dim\mathbf{S}_+=2$.

定理 4.2.2 $\mathbf{S}$ 的完全 Lagrangian 子流形有且仅有 0- 级和 1- 级的.

证明 由引理 4.4 和文 [1] 的定理 5 知.

因为 $\dim\mathbf{S}=4$, 所以 $\mathbf{S}$ 与 C^4 线性同构. 因此我们可以和用 C^4 的单位基向量

$$e^1=(1,0,0,0),\ e^2=(0,1,0,0),\ e^3=(0,0,1,0),\ e^4=(0,0,0,1)$$

来表示 $\mathbf{S}$, 即 $\mathbf{S}=\text{span}\{e^1,e^2,e^3,e^4\}$. 设 $\mathbf{f}\in\mathbf{S}, f\in D(T_{\max})$, 则可以如下选取 $\mathbf{f}$ 的坐标:

$$\begin{aligned}\mathbf{f}=&(f(0),p(0)f'(0),[f\phi](\infty),[f\psi](\infty))\\=&f(0)e^1+p(0)f^1(0)e^2+[f\phi](\infty)e^3+[f\psi](\infty)e^4,\end{aligned}$$

且有以下结论:

定理 4.2.3 对于 $\forall\mathbf{f},\mathbf{g}\in\mathbf{S}$ 有

$$[\mathbf{f}:\mathbf{g}]=\mathbf{f}H\mathbf{g}^*,$$

其中

$$H=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}.$$

证明 由 $\mathbf{S}$ 的辛形式的定义和文 [11] 有

$$\begin{aligned}[\mathbf{f}:\mathbf{g}]=&[fg]_0^\infty=[fg](\infty)-[fg](0)=[fg](\infty)\left[\bar{\phi}\psi\right](\infty)-[fg](0)\\=&\begin{vmatrix}[f\phi](\infty)&[f\psi](\infty)\\ [\bar{g}\phi](\infty)&[\bar{g}\psi](\infty)\end{vmatrix}-[fg](0)\\=&[\bar{g}\psi](\infty)[f\phi](\infty)-[\bar{g}\phi](\infty)[f\psi](\infty)+p(0)f'(0)\bar{g}(0)-p(0)f(0)\bar{g}'(0)\\=&\mathbf{f}H\mathbf{g}^*.\end{aligned}$$

定理 4.2.4 $\mathbf{S}_-=\operatorname{span}\{e^1,e^2\},\mathbf{S}_+=\operatorname{span}\{e^3,e^4\}$.

证明 若 $\mathbf{f}\in\mathbf{S}_-$, 则 $\mathbf{f}\in\mathbf{S}, f\in D(T_{\max})$ 且 $[f\phi](\infty)=[f\psi](\infty)=0$. 因此

$$\mathbf{f}=f(0)e^1+p(0)f^1(0)e^2\in\operatorname{span}\{e^1,e^2\}.$$

若 $\mathbf{f}\in\operatorname{span}\{e^1,e^2\}$, 则 $\mathbf{f}=f(0)e^1+p(0)f^1(0)e^2$. 因此 $[f\varphi](\infty)=[f\psi](\infty)=0$. 从而 $\mathbf{f}\in\mathbf{S}_-$. 这就证明了 $\mathbf{S}_-=\operatorname{span}\{e^1,e^2\}$. 类似可证 $\mathbf{S}_+=\operatorname{span}\{e^3,e^4\}$.

定理 4.2.5 $\mathbf{L}$ 是 $\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形 $\Longleftrightarrow$, $\exists a_1,a_2,b_1,b_2\in C$, 使得 $\mathbf{L}=\operatorname{span}\{a_1e^1+a_2e^2,b_1e^3+b_2e^4\}$, 且满足

(1) a_1 与 a_2 不全为零, b_1 与 b_2 不全为零;

(2) $\begin{vmatrix}a_1&a_2\\\bar{a}_1&\bar{a}_2\end{vmatrix}=\begin{vmatrix}b_1&b_2\\\bar{b}_1&\bar{b}_2\end{vmatrix}=0.$

证明 ($\Longleftarrow$) 对于任意 $\mathbf{f},\mathbf{g}\in\mathbf{L}$, 则存在 $\alpha_1,\alpha_2,\beta_1,\beta_2\in C$, 使得

$$\begin{aligned}\mathbf{f}=&\alpha_1(a_1e^1+a_2e^2)+\beta_1(b_1e^3+b_2e^4)=\alpha_1a_1e^1+\alpha_1a_2e^2+\beta_1b_1e^3+\beta_1b_2e^4\\\mathbf{g}=&\alpha_2(a_1e^1+a_2e^2)+\beta_2(b_1e^3+b_2e^4)=\alpha_2a_1e^1+\alpha_2a_2e^2+\beta_2b_1e^3+\beta_2b_2e^4,\end{aligned}$$

则由定理 4.2.3 和条件 (2) 得

$$[\mathbf{f}:\mathbf{g}]=(\alpha_1a_1,\alpha_1a_2,\beta_1b_1,\beta_1b_2)H(\alpha_2a_1,\alpha_2a_2,\beta_2b_1,\beta_2b_2)^*=0$$

故 $[\mathbf{L}:\mathbf{L}]=0$, 即 $\mathbf{L}$ 是 $\mathbf{S}$ 的 *Lagrangian* 子流形. 由 (1) 知, $\dim\mathbf{L}=2$. 因此, 由引理 4.5 知, $\mathbf{L}$ 是 $\mathbf{S}$ 的完全 Lagrangian 子流形. 由 (1) 知,

$$\dim\mathbf{L}\cap\mathbf{S}_-=\dim\mathbf{L}\cap\mathbf{S}_+=1,$$

所以 $\mathbf{L}$ 是 0- 级的.

($\Longleftarrow$) 因为 $\mathbf{L}$ 是 $\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形, 所以

$$\dim\mathbf{L}=2,\dim\mathbf{L}\cap\mathbf{S}_-=\dim\mathbf{L}\cap\mathbf{S}_+=1,[\mathbf{L}:\mathbf{L}]=0.$$

因此 $\exists a_1,a_2,b_1,b_2\in C$, 使得

$$\mathbf{L}=\operatorname{span}\{a_1e^1+a_2e^2,b_1e^3+b_2e^4\}$$

易证明 $\mathbf{L}$ 满足条件 (1) 与 (2).

推论 4.2.6 $\mathbf{L}=\operatorname{span}\{a_1e^1+a_2e^2,b_1e^3+b_2e^4\}$ 是 $\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形 $\Longleftrightarrow\exists a_1,a_2,b_1,b_2\in C$, 使得 $\mathbf{L}$ 满足以下条件之一:

(1) 当 a_1 与 a_2 中仅有一个为零, b_1 与 b_2 中仅有一个为零时, 不妨设 $a_2=b_2=0$, 则

$$\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|f'(0)=0,[f\psi](\infty)=0\}.$$

(2) 当 a_1,a_2,b_1,b_2 中只有一个为零时, 不妨设 $b_2=0$, 则

$$\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|a_2f(0)=a_1p(0)f'(0),[f\psi](\infty)=0\}.$$

且 $\begin{vmatrix}a_1 & a_2\\ \bar{a}_1 & \bar{a}_2\end{vmatrix}=0.$

(3) 当 a_1,a_2,b_1,b_2 全非零时, 则

$$\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|a_2f(0)=a_1p(0)f'(0),b_2[f\varphi](\infty)=b_1[f\psi](\infty)\}.$$

且 $\begin{vmatrix}a_1 & a_2\\ \bar{a}_1 & \bar{a}_2\end{vmatrix}=\begin{vmatrix}b_1 & b_2\\ \bar{b}_1 & \bar{b}_2\end{vmatrix}=0.$

例 4.2.1

$$\mathbf{L}=\operatorname{span}\{e^1+e^2,e^3+e^4\}=\{\bar{f}\in S|f(0)=p(0)f'(0),[f\phi](\infty)=[f\psi](\infty)\}$$

是 $\mathbf{S}$ 的一个 0- 级完全 Lagrangian 子流形.

类似可得如下结论:

定理 4.2.7 $\mathbf{L}$ 是 $\mathbf{S}$ 的 1- 级完全 Lagrangian 子流形 $\Longleftrightarrow\exists a_i,b_i\in C(i=1,2,3,4)$, 使得

$$\mathbf{L}=\operatorname{span}\{a_1e^1+a_2e^2+a_3e^3+a_4e^4,b_1e^1+b_2e^2+b_3e^3+b_4e^4\}$$

且满足:

(1) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} \neq 0;$

(2) $\begin{vmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{a}_3 & \bar{a}_4 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} b_3 & b_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}.$

推论 4.2.8 **L** 是 **S** 的 1- 级完全 Lagrangian 子流形 $\Longleftrightarrow \exists a_i, b_i \in C(i=1,2,3,4)$, 使得

$$\mathbf{L} = \{\mathbf{f} \in \mathbf{S} | \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} f(0) \\ p(0)f'(0) \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}^{-1} \begin{pmatrix} [f\phi](\infty) \\ [f\psi](\infty) \end{pmatrix}\},$$

且满足:

(1) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} \neq 0;$

(2) $\begin{vmatrix} a_1 & a_2 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{a}_3 & \bar{a}_4 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} b_3 & b_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ \bar{b}_1 & \bar{b}_2 \end{vmatrix} = \begin{vmatrix} a_3 & a_4 \\ \bar{b}_3 & \bar{b}_4 \end{vmatrix}.$

注 从定理 4.2.5 、推论 4.2.6 、定理 4.2.7 和推论 4.2.8 可以看出， **S** 的 0- 级与 1- 级完全 Lagrangian 子流形都可以用边条件来描述， 0- 级完全 Lagrangian 子流形的边条件是分离的，而 1- 级完全 Lagrangian 子流形的边条件不是分离的，是耦合的. 从这个角度也说明了， Weyl-Titchmarsh 域是不完全的，仅给出了 0- 级自伴域的描述；而 Cao 域是自伴域的完全描述.

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Complex Symplectic Geometry Characterization for Self-adjoint Domains of Second Order Ordinary Differential Operators

Abstract Let $l(y) = -(py')' + qy$ be a real symmetric differential expression defined on interval I. In $L^2(I)$, we classify the self-adjoint domains generated by $l(y)$ and give the complete characterization for k–grade self-adjoint domains with complex symplectic geomety.

Key words Differential operator; Self-adjoint domain; Symplectic geomety; Submanifold

MR(1991) Subject Classification 34B05, 34L05, 47B05, 58F05

Chinese Library Classification O175.3

高阶常型微分算子自伴域的辛几何刻划

摘 要 考虑高阶常型实系数微分算子 $L(y)=\sum_{k=0}^{n}(p_{n-k}y^{(k)})^{(k)}$ $(x\in[a,b])$. 利用辛几何，对 $l(y)$ 的自伴域进行了分类，给出了 $l(y)$ 自伴域是 k- 级的充要条件 $(0\le k\le n)$.

关键词 微分算子；自伴域；辛几何；子流形
MR(1991) 主题分类 34B05, 34L05, 47B05, 58F05
中图分类 O175.3

Complex Symplectic Geometry Characterization for Self-adjoint Domains of 2n–th Order Non-singular Differential Operators

Abstract Let $L(y)=\sum_{k=0}^{n}(p_{n-k}y^{(k)})^{(k)}$ be a real symmetric differential expression defined on interval $I=[a,b]$. In $L^2(I)$, we classify the self-adjoint domains generated by $l(y)$ and give the complete characterization for self-adjoint domains with complex symplectic geomety.

Keywords Differential operator; Self-adjoint domain; Symplectic geomety; Submanifold
MR(1991) Subject Classification 34B05, 34L05, 47B05, 58F05
Chinese Library Classification O175.3

1 引言

设
$$l(y)=\sum_{k=0}^{n}(p_{n-k}y^{(k)})^{(k)}$$
是 $I=[a,b]$ 上的 2n 阶实系数微分算式，$p_{n-k}\in C^k[a,b](k=0,1,\cdots,n)$, $p_0(x)$ 在 I 上恒大于零.

由 [11] 知，$l(y)$ 的亏指数必为 (2,2) 且必可生成自伴算子. 如何去描述 $l(y)$ 的自伴域呢？ 1954 年， E.A.Coddington 在 [13] 中给出

了这个问题的完全解答. 同一时期, M.A. Naimark 在 [4] 中给出了由"拟导数"定义的对称微分算子自伴域的完全刻划. 1962, W.N.Everitt 在 [14] 中应用微分方程 $l(y)=\lambda y$ 的解给出了自伴域的描述. 1999 年, W.N.Everitt 和 L.Markus 在文 [1] 中利用辛几何也给出了微分算子 $l(y)$ 自伴扩张的完全刻划.

微分算子理论是当代量子力学的数学支柱,是解决数学物理方程以及大量科学技术应用问题的重要数学工具. 微分算子自伴扩张问题是微分算子理论的基础问题之一, 受到大家的广泛关注, 如文 [1–14] 等. 以前的大部分研究工作是利用分析、算子等方法对自伴域进行描述, 并未对自伴域进行分类.

本文对 $2n$ 阶常型实系数对称微分算子, 利用辛几何的方法, 对其自伴域进行了分类, 即分为 $n+1$ 类, 同时也给出了自伴域是 k– 级的充分必要条件 $(0\le k\le n)$.

2 预备知识

定义 2.1 一个复的辛空间 $\mathbf{S}$ 是一个复的线性空间, 且带有一个辛形式 $[:]$, 即

(1) $[:]$ 是一个半双线性型,

$$u,v\to[u:v],\mathbf{S}\times\mathbf{S}\to C,[c,u+c_2v:w]=c_1[u:w]+c_2[v:w],$$

(2) $[:]$ 是一个反 Hermitian 型,

$$[u:v]=-\overline{[v:u]},[u:c_1v+c_2w]=\overline{c_1}[u:v]+\overline{c_2}(u:w],$$

(3) $[:]$ 是非退化的,

$$[u:\mathbf{S}]=0\Longrightarrow u=0,$$

对 $\forall u_1v_1w\in\mathbf{S},\forall c_1,c_2\in C$.

定义 2.2 复辛空间 $\mathbf{S}$ 的一个线性流形 $\mathbf{L}$ 被称为是 Lagrangian 的, 若 $[\mathbf{L}:\mathbf{L}]=0$, 即, 对 $\forall u,v\in\mathbf{L}$ 有 $[u:v]=0$.

$\mathbf{S}$ 的一个 Lagrangian 子流形 $\mathbf{L}$ 被称为是完全的, 若 $u\in\mathbf{S}$ 且 $[u:\mathbf{L}]=0\Longrightarrow u\in\mathbf{L}$.

定义 2.3 设 $\mathbf{S}$ 是一个复辛空间. 若 $\mathbf{S}_-$ 和 $\mathbf{S}_+$ 是 $\mathbf{S}$ 的线性子流形, 且满足:

(1) $\mathbf{S}=\mathrm{span}\{\mathbf{S}_-,\mathbf{S}_+\}$;

(2) $[\mathbf{S}_- : \mathbf{S}_+] = 0$;

则称 $\mathbf{S}_-$ 和 $\mathbf{S}_+$ 在 $\mathbf{S}$ 中是辛正交互补，记作 $\mathbf{S} = \mathbf{S}_- \oplus \mathbf{S}_+$.

有关辛几何的概念详细见文 [1].

设

$$l(y) = \sum_{k=0}^{n} (p_{n-k} y^{(k)})^{(k)}$$

是 $I = [a, b]$ 上的 2n 阶实系数微分算式，$p_{n-k} \in C^k[a, b] (k = 0, 1, \cdots, n)$，$p_0(x)$ 在 I 上恒大于零.

由 $l(y)$ 生成的最大算子与最小算子定义如下：

$T_{\max}(y) = l(y), y \in D(T_{\max}) = \{y \in L^2(I) | y^{[k]} \in AC(I), k = 1, 2, \cdots, 2n-1, y^{[2n]} \in L^2(I)\}$;

$T_{\min}(y) = l(y), y \in D(T_{\min}) = \{y \in D(T_{\max}) | y^{[k]}(a) = y^{[k]}(b) = 0, k = 0, 1, \cdots, 2n-1\}$;

其中 $y^{[k]}$ 是 y 的拟导数，即

$y^{[k]} = \frac{d^k y}{dx^k}$，$0 \le k \le n-1$,

$y^{[n]} = p_0 \frac{d^n y}{dx^n}$,

$y^{[n+k]} = p_k \frac{d^{n-k}}{dx^{n-k}} - \frac{d}{dx}(y^{[n+k-1]})$，$1 \le k \le n$.

由常微分算子理论知，$T_{\min}$ 和 $T_{\max}$ 是闭线性算子，且 $T_{\max}^* = T_{\min}, T_{\min}^* = T_{\max}^*$.

令

$$\mathbf{S} = D(T_{\max}) / D(T_{\min}),$$

在 $\mathbf{S}$ 中定义辛形式 $[:]$ 为：

$$[\mathbf{f} : \mathbf{g}] = [f + D(T_{\min}) : g + D(T_{\min})] = [fg]_a^b, \quad \forall f, g \in D(T_{\max})$$

其中 $\mathbf{f} = f + D(T_{\min}), \mathbf{g} = g + D(T_{\min}) \in \mathbf{S}$，$[fg]_a^b$ 是 f 与 g 的契合式. 若令

$$[fg]_a^b = [f : g],$$

则由文 [1],[4] 和 [11] 知， $D(T_{\min})$ 可表示为

$$D(T_{\min}) = \{f \in D(T_{\max}) | [f : D(T_{\max})] = 0\},$$

易知 $T_{\min}$ 是一个对称算子.

由文 [1],[4] 和 [11] 可得如下引理：

引理 2.1 $\mathbf{S} = D(T_{\max}) / D(T_{\min})$ 是一个 $4n$ 维复辛空间.

引理 2.2 $\mathbf{S}=\mathbf{S}_-\oplus\mathbf{S}_+$, 其中

$$\mathbf{S}_-=\{\mathbf{f}\in\mathbf{S}|f^{[k]}(b)=0,k=0,1,\cdots,2n-1\},$$
$$\mathbf{S}_+=\{\mathbf{f}\in\mathbf{S}|f^{[k]}(a)=0,k=0,1,\cdots,2n-1\},$$

且 $\dim\mathbf{S}_-=\dim\mathbf{S}_+=2n$.

引理 2.3 (平衡相交原理) 若 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形, 则

$$\begin{aligned}0&\le\frac{1}{2}\dim\mathbf{S}_--\dim\mathbf{L}\cap\mathbf{S}_-=\frac{1}{2}\dim\mathbf{S}_+-\dim\mathbf{L}\cap\mathbf{S}_+\\&\le\frac{1}{2}\min\{\dim\mathbf{S}_-,\dim\mathbf{S}_+\},\end{aligned}$$

即

$$0\le n-\dim\mathbf{L}\cap\mathbf{S}_-=n-\dim\mathbf{L}\cap\mathbf{S}_+\le n.$$

定义 2.4 设 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形, 令

$$k=n-\dim\mathbf{L}\cap\mathbf{S}_-=n-\dim\mathbf{L}\cap\mathbf{S}_+.$$

则称 $\mathbf{L}$ 是 k- 级的, 也称 $D(T_{\mathbf{L}})$ 是 k- 级的.

引理 2.4 (GKN 定理) (1) $T_{\min}$ 有自伴扩张 $\Longleftrightarrow$ $\mathbf{S}$ 有完全 Lagrangian 子流形.

(2) $\mathbf{S}$ 的 Lagrangian 子流形 $\mathbf{L}$ 是完全的 $\Longleftrightarrow$ $\dim\mathbf{L}=\frac{1}{2}\dim\mathbf{S}=2n$.

(3) 若 T 是 $T_{\min}$ 的一个自伴扩张, 自伴域为 $D(T)$, 则 S 有唯一的完全 Lagrangian 子流形 $\mathbf{L_T}$ 与其对应, 使得

$$\mathbf{L_T}=D(T)/D(T_{\min}).$$

(4) 若 $\mathbf{L}$ 是 $\mathbf{S}$ 的一个完全 Lagrangian 子流形, 则 $T_{\min}$ 有唯一的自伴扩张 $T_{\mathbf{L}}$ 与其对应, 使得

$$D(T_{\mathbf{L}})=c_1f_1+c_2f_2+\cdots+c_{2n}f_{2n}+D(T_{\min}),$$

其中 $\mathbf{f_1},\mathbf{f_2},\cdots,\mathbf{f_{2n}}$ 是 $\mathbf{L}$ 的一个基, $f_1,f_2,\cdots,f_{2n}\in D(T_{\max})$,$c_1,c_2,\cdots,c_{2n}$ 是任意复数.

注 由引理 2.4 知, 讨论 $T_{\min}$ 的自伴扩张问题等价于讨论复辛空间 $\mathbf{S}=D(T_{\max})/D(T_{\min})$ 的完全 Lagrangian 子流形. 因此, 对 $\mathbf{S}$ 的完全

Lagrangian 子流形进行的分类和描述就等价于对 $l(y)$ 的自伴域进行的分类和描述.

3 结论与证明

定理 3.1 S 的完全 Lagrangian 子流形有且仅有 0- 级, 1- 级, ⋯, 和 n- 级的.

证明 由引理 2.3 与文 [1] 的定理 5 知.

因为 $\dim \mathbf{S} = 4n$, 所以 S 与 C^{4n} 线性同构. 因此我们可以利用 C^{4n} 的单位基向量

$$e^1 = (1,0,0,\cdots,0),\ e^2 = (0,1,0,\cdots,0),\cdots,\ e^{2n} = (\overbrace{0,\cdots,0,1}^{2n},0,\cdots,0),$$

$$f^1 = (\overbrace{0,\cdots,0,1}^{2n+1},0,\cdots,0), f^2 = (\overbrace{0,\cdots,0,1}^{2n+2},0,\cdots,0),\cdots,f^{2n} = (0,\cdots,0,1)$$

来表示 S, 即 $\mathbf{S} = \mathrm{span}\{e^1,e^2,\cdots,e^{2n},f^1,f^2,\cdots,f^{2n}\}$. 设 $\mathbf{f} \in \mathbf{S}$, 则可以如下选取 $\mathbf{f}$ 的坐标:

$$\begin{aligned}\mathbf{f} =&(f(a),f^{[1]}(a),\cdots,f^{[2n-1]}(a),f(b),f^{[1]}(b),\cdots,f^{[2n-1]}(b))\\ =&f(a)e^1 + f^{[1]}(a)e^2 + \cdots + f^{[2n-1]}(a)e^{2n}\\ &+ f(b)f^1 + f^{[1]}(b)f^2 + \cdots + f^{[2n-1]}(b)f^{2n},\end{aligned} \tag{3.1}$$

且有结论:

定理 3.2 对于 $\forall \mathbf{f},\mathbf{g} \in \mathbf{S}$, 有

$$[\mathbf{f}:\mathbf{g}] = \mathbf{f}H\mathbf{g}^*,$$

其中

$$H = \begin{pmatrix} 0 & -H_1 & 0 & 0 \\ H_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_1 \\ 0 & 0 & -H_1 & 0 \end{pmatrix},\quad H_1 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ & \cdots & \cdots & \\ 1 & \cdots & 0 & 0 \end{pmatrix}_{n\times n}$$

证明 由 S 的辛形式的含义和文 [4, *p*.180] 的 *Lagrange* 公式有:

$$\begin{aligned}[\mathbf{f}:\mathbf{g}] =&[fg]_a^b = \langle l(f),g\rangle - \langle f,l(g)\rangle\\ =&\sum_{k=1}^{n}\{f^{[k-1]}\bar{g}^{[2n-k]} - f^{[2n-k]}\bar{g}^{[k-1]}\}|_a^b\\ =&\mathbf{f}H\mathbf{g}^*.\end{aligned}$$

定理 3.3 $\mathbf{S}_- = \text{span}\{e^1, e^2, \cdots, e^{2n}\}, \mathbf{S}_+ = \text{span}\{f^1, f^2, \cdots, f^{2n}\}$.

证明 下面证明 $\mathbf{S}_- = \text{span}\{e^1, e^2, \cdots, e^{2n}\}$.

对于 $\forall \mathbf{f} \in \mathbf{S}_-$, 则 $\mathbf{f} \in \mathbf{S}, f \in D(T_{\max})$ 且 $f^{[k]}(b) = 0, k = 0, 1, \cdots, 2n-1$. 因此

$$\mathbf{f} = f(a)e^1 + f^{[1]}(a)e^2 + \cdots + f^{[2n-1]}(a)e^{2n} \in \text{span}\{e^1, e^2, \cdots, e^{2n}\}.$$

故 $\mathbf{S}_- \subseteq \text{span}\{e^1, e^2, \cdots, e^{2n}\}$.

若 $\mathbf{f} \in \text{span}\{e^1, e^2, \cdots, e^{2n}\}$, 则 $\mathbf{f} = f(a)e^1 + f^{[1]}(a)e^2 + \cdots + f^{[2n-1]}(a)e^{2n}$, 即 $f^{[k]}(b) = 0, k = 0, 1, \cdots, 2n-1$. 因此 $\text{span}\{e^1, e^2, \cdots, e^{2n}\} \subseteq \mathbf{S}_-$. 于是 $\mathbf{S}_- = \text{span}\{e^1, e^2, \cdots, e^{2n}\}$.

类似可证 $\mathbf{S}_+ = \text{span}\{f^1, f^2, \cdots, f^{2n}\}$.

定理 3.4 $\mathbf{L}$ 是 $\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形 $\Longleftrightarrow \exists a_{ij}, b_{ij} \in C, i = 1, 2, \cdots, n, j = 1, 2, \cdots, 2n$,, 使得

$$\begin{aligned}\mathbf{L} = \text{span}\{&a_{11}e^1 + a_{12}e^2 + \cdots + a_{1,2n}e^{2n}, \cdots, a_{n1}e^1 + a_{n2}e^2 + \cdots + a_{n,2n}e^{2n}, \\ &b_{11}f^1 + b_{12}f^2 + \cdots + b_{1,2n}f^{2n}, \cdots, b_{n1}f^1 + b_{n2}f^2 + \cdots + b_{n,2n}f^{2n}\} \quad (3.2)\end{aligned}$$

且满足

(1) 秩 $A_n =$ 秩 $B_n = n$;

(2) $\alpha_i H \alpha_j^* = 0, 1 \le i, j \le 2n$;

其中 $A_n = (a_{ij})_{n\times 2n}, B_n = (b_{ij})_{n\times 2n}$,

$$\alpha_i = \begin{cases} (a_{i1}, \cdots, a_{i,2n}, 0, \cdots, 0), & 1 \le i \le n, \\ (0, \cdots, 0, b_{i1}, \cdots, b_{i,2n}), & n+1 \le i \le 2n. \end{cases}$$

证明($\Longleftarrow$) 对于任意 $\mathbf{f}, \mathbf{g} \in \mathbf{L}$, 则存在 $s_{1i}, s_{2i}, t_{1i}, t_{2i} \in C, i = 1, 2, \cdots, n$, 使得

$$\begin{aligned}\mathbf{f} &= \sum_{i=1}^n s_{1i}(a_{i1}e^1 + a_{i2}e^2 + \cdots + a_{i,2n}e^{2n}) + \sum_{i=1}^n t_{1i}(b_{i1}f^1 + b_{i2}f^2 + \cdots + b_{i,2n}f^{2n}) \\ &= (\sum_{i=1}^n s_{1i}a_{i1})e^1 + \cdots + (\sum_{i=1}^n s_{1i}a_{i,2n})e^{2n} + (\sum_{i=1}^n t_{1i}b_{i1})f^1 + \cdots + (\sum_{i=1}^n t_{1i}b_{i,2n})f^{2n} \\ \mathbf{g} &= \sum_{i=1}^n s_{2i}(a_{i1}e^1 + a_{i2}e^2 + \cdots + a_{i,2n}e^{2n}) + \sum_{i=1}^n t_{2i}(b_{i1}f^1 + b_{i2}f^2 + \cdots + b_{i,2n}f^{2n}) \\ &= (\sum_{i=1}^n s_{2i}a_{i1})e^1 + \cdots + (\sum_{i=1}^n s_{1i}a_{i,2n})e^{2n} + (\sum_{i=1}^n t_{2i}b_{i1})f^1 + \cdots + (\sum_{i=1}^n t_{2i}b_{i,2n})f^{2n}\end{aligned}$$

则由定理 3.2 和条件 (2) 得

$$\begin{aligned}[\mathbf{f}:\mathbf{g}]=&(\sum_{i=1}^{n}s_{1i}a_{i1},\cdots,\sum_{i=1}^{n}s_{1i}a_{i,2n},\sum_{i=1}^{n}t_{1i}b_{i1},\cdots,\sum_{i=1}^{n}t_{1i}b_{i,2n})H\\&(\sum_{i=1}^{n}s_{2i}a_{i1},\cdots,\sum_{i=1}^{n}s_{1i}a_{i,2n},\sum_{i=1}^{n}t_{2i}b_{i1},\cdots,\sum_{i=1}^{n}t_{2i}b_{i,2n})^*\\=&0.\end{aligned}$$

故 $[\mathbf{L}:\mathbf{L}]=0$ ，即 $\mathbf{L}$ 是 $\mathbf{S}$ 的 *Lagrangian* 子流形. 由 (1) 知, $\dim\mathbf{L}=2n$. 因此，由引理 2.4 知， $\mathbf{L}$ 是 $\mathbf{S}$ 的完全 Lagrangian 子流形. 由 (1) 知

$$\dim\mathbf{L}\cap\mathbf{S}_-=\dim\mathbf{L}\cap\mathbf{S}_+=n,$$

所以 $\mathbf{L}$ 是 0- 级的.

($\Longleftarrow$) 因为 $\mathbf{L}$ 是 $\mathbf{S}$ 的 0- 级完全 Lagrangian 子流形，所以

$$\dim\mathbf{L}=2n,\dim\mathbf{L}\cap\mathbf{S}_-=\dim\mathbf{L}\cap\mathbf{S}_+=n,[\mathbf{L}:\mathbf{L}]=0.$$

因此 $\exists a_{ij},b_{ij}\in C,i=1,2,\cdots,n,j=1,2,\cdots,2n,$, 使得

$$\begin{aligned}\mathbf{L}=\text{span}\{&a_{11}e^1+a_{12}e^2+\cdots+a_{1,2n}e^{2n},\cdots,a_{n1}e^1+a_{n2}e^2+\cdots+a_{n,2n}e^{2n},\\&b_{11}f^1+b_{12}f^2+\cdots+b_{1,2n}f^{2n},\cdots,b_{n1}f^1+b_{n2}f^2+\cdots+b_{n,2n}f^{2n}\}\end{aligned}$$

易证明 $\mathbf{L}$ 满足条件 (1) 与 (2).

推论 3.5 $\mathbf{L}$ 是 $\mathbf{S}$ 的 0- 级完全 Langrangian 子流形 $\Longleftrightarrow\exists a_{ij},b_{ij}\in C,i=1,2,\cdots,n,j=1,2,\cdots,2n,$, 使得

$$\begin{aligned}\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|\exists s_i,t_i\in C,i=1,2,\cdots,n,\\&(f(a),f^{[1]}(a),\cdots,f^{[2n-1]}(a))^T=A_n^T(s_1,s_2,\cdots,s_n)^T,\\&(f(b),f^{[1]}(b),\cdots,f^{[2n-1]}(b))^T=B_n^T(t_1,t_2,\cdots,t_n)^T\}\end{aligned}$$

且满足

(1) 秩 $A_n=$ 秩 $B_n=n$;

(2) $\alpha_iH\alpha_j^*=0,1\le i,j\le 2n$;

其中 $A_n=(a_{ij})_{n\times 2n},B_n=(b_{ij})_{n\times 2n}$,

$$\alpha_i=\begin{cases}(a_{i1},\cdots,a_{i,2n},0,\cdots,0), & 1\le i\le n,\\(0,\cdots,0,b_{i1},\cdots,b_{i,2n}), & n+1\le i\le 2n.\end{cases}$$

证明 ($\Longrightarrow$) 设 $\mathbf{f}\in\mathbf{L}$, 则由定理 3.4 知，存在 $s_i,t_i\in C,i=$

$1,2,\cdots,n$, 使得

$$
\begin{aligned}
\mathbf{f} =& \sum_{i=1}^{n} s_i(a_{i1}e^1 + a_{i2}e^2 + \cdots + a_{i,2n}e^{2n}) + \sum_{i=1}^{n} t_i(b_{i1}f^1 + b_{i2}f^2 + \cdots + b_{i,2n}f^{2n}) \\
=& (\sum_{i=1}^{n} s_i a_{i1})e^1 + \cdots + (\sum_{i=1}^{n} s_i a_{i,2n})e^{2n} + (\sum_{i=1}^{n} t_i b_{i1})f^1 + \cdots + (\sum_{i=1}^{n} t_i b_{i,2n})f^{2n} \quad (3.3)
\end{aligned}
$$

由 (3.1) 式得,

$$
\begin{cases}
\sum_{i=1}^{n} s_i a_{i1} = f(a) \\
\cdots \\
\sum_{i=1}^{n} s_i a_{i,2n} = f^{[2n-1]}(a)
\end{cases}
,\qquad
\begin{cases}
\sum_{i=1}^{n} t_i b_{i1} = f(b) \\
\cdots \\
\sum_{i=1}^{n} t_i b_{i,2n} = f^{[2n-1]}(b)
\end{cases}
\quad (3.4)
$$

即

$$(f(a), f^{[1]}(a), \cdots, f^{[2n-1]}(a))^T = A_n^T(s_1, s_2, \cdots, s_n)^T, \quad (3.5)$$

$$(f(b), f^{[1]}(b), \cdots, f^{[2n-1]}(b))^T = B_n^T(t_1, t_2, \cdots, t_n)^T\}. \quad (3.6)$$

($\Longleftarrow$) 对于任意 $\mathbf{f} \in \mathbf{L}$, 由 (3.5) 和 (3.6) 知, (3.4) 成立. 由 (3.1) 式知, (3.3) 成立, 故 (3.2) 成立. 由定理 3.4 知, 结论成立.

例 1 $\mathbf{L} = \mathrm{span}\{e^1 + e^2, e^3 + e^4, \cdots, e^{2n-1} + e^{2n}, f^1 + f^2, f^3 + f^4, \cdots, f^{2n-1} + f^{2n}\} = \{\mathbf{f} \in \mathbf{S} | f^{[k]}(a) = f^{[k+1]}(a), f^{[k]}(b) = f^{[k+1]}(b), k = 0, 1, \cdots, 2n-2\}$ 是 $\mathbf{S}$ 的一个 0- 级完全 Lagrangian 子流形.

定理 3.6 $\mathbf{L}$ 是 $\mathbf{S}$ 的 1- 级完全 Lagrangian 子流形 $\Longleftrightarrow \exists a_{ij}, b_{ij}, c_{kj}, d_{kj} \in C, i = 1, 2, \cdots, n-1, j = 1, 2, \cdots, 2n, k = 1, 2$, 使得

$$
\begin{aligned}
\mathbf{L} = \mathrm{span}\{& a_{11}e^1 + a_{12}e^2 + \cdots + a_{1,2n}e^{2n}, \cdots, a_{n-1,1}e^1 + a_{n-1,2}e^2 + \cdots + a_{n-1,2n}e^{2n}, \\
& b_{11}f^1 + b_{12}f^2 + \cdots + b_{1,2n}f^{2n}, \cdots, b_{n-1,1}f^1 + b_{n-1,2}f^2 + \cdots + b_{n-1,,2n}f^{2n} \\
& c_{11}e^1 + c_{12}e^2 + \cdots + c_{1,2n}e^{2n} + d_{11}f^1 + d_{12}f^2 + \cdots + d_{1,2n}f^{2n}, \\
& c_{21}e^1 + c_{22}e^2 + \cdots + c_{2,2n}e^{2n} + d_{21}f^1 + d_{22}f^2 + \cdots + d_{2,2n}f^{2n}\}
\end{aligned}
$$

且满足

(1) 秩 $A_{n+1} =$ 秩 $B_{n+1} = n+1$;

(2) $\alpha_i H \alpha_j^* = 0, 1 \le i, j \le 2n$;

其中

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,2n} \\ & \cdots & \cdots & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,2n} \\ c_{11} & c_{12} & \cdots & c_{1,2n} \\ c_{21} & c_{22} & \cdots & c_{2,2n} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,2n} \\ & \cdots & \cdots & \\ b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,2n} \\ d_{11} & d_{12} & \cdots & d_{1,2n} \\ d_{21} & d_{22} & \cdots & d_{2,2n} \end{pmatrix}.$$

$$\alpha_i=\begin{cases} (a_{i1},\cdots,a_{i,2n},0,\cdots,0), & 1\le i\le n-1, \\ (0,\cdots,0,b_{i1},\cdots,b_{i,2n}), & n\le i\le 2n-2, \\ (c_{i1},\cdots,c_{i,2n},d_{i1},\cdots,d_{i,2n}), & 2n-1\le i\le 2n. \end{cases}$$

证明 类似于定理 3.4 的证明.

推论 3.7 **L** 是 **S** 的 1- 级完全 Langrangian 子流形 $\Longleftrightarrow \exists a_{ij},b_{ij},c_{kj},d_{kj}\in C, i=1,2,\cdots,n-1, j=1,2,\cdots,2n, k=1,2$, 使得

$$\begin{aligned}\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|\exists s_i,t_i,u_j\in C, i=1,2,\cdots,n-1, j=1,2,\\ (f(a),f^{[1]}(a),\cdots,f^{[2n-1]}(a))^T=A_{n+1}^T(s_1,s_2,\cdots,s_{n-1},u_1,u_2)^T,\\ (f(b),f^{[1]}(b),\cdots,f^{[2n-1]}(b))^T=B_{n+1}^T(t_1,t_2,\cdots,t_{n-1},u_1,u_2)^T\}.\end{aligned}$$

且满足

(1) 秩 $A_{n+1}=$ 秩 $B_{n+1}=n+1$;

(2) $\alpha_i H\alpha_j^*=0, 1\le i,j\le 2n$;

其中

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,2n} \\ & \cdots & \cdots & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,2n} \\ c_{11} & c_{12} & \cdots & c_{1,2n} \\ c_{21} & c_{22} & \cdots & c_{2,2n} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,2n} \\ & \cdots & \cdots & \\ b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,2n} \\ d_{11} & d_{12} & \cdots & d_{1,2n} \\ d_{21} & d_{22} & \cdots & d_{2,2n} \end{pmatrix}.$$

$$\alpha_i=\begin{cases} (a_{i1},\cdots,a_{i,2n},0,\cdots,0), & 1\le i\le n-1, \\ (0,\cdots,0,b_{i1},\cdots,b_{i,2n}), & n\le i\le 2n-2, \\ (c_{i1},\cdots,c_{i,2n},d_{i1},\cdots,d_{i,2n}), & 2n-1\le i\le 2n. \end{cases}$$

定理 3.8 **L** 是 **S** 的 k- 级完全 Lagrangian 子流形 $\Longleftrightarrow \exists a_{ij},b_{ij},c_{lj},d_{lj}\in$

$C, i=1,2,\cdots,n-k, j=1,2,\cdots,2n, l=1,2,\cdots,2k$, 使得

$$= \text{span}\{a_{11}e^1+a_{12}e^2+\cdots+a_{1,2n}e^{2n},\cdots,a_{n-k,1}e^1+a_{n-k,2}e^2+\cdots+a_{n-k,2n}e^{2n},\\ b_{11}f^1+b_{12}f^2+\cdots+b_{1,2n}f^{2n},\cdots,b_{n-k,1}f^1+b_{n-k,2}f^2+\cdots+b_{n-k,,2n}f^{2n}\\ c_{11}e^1+c_{12}e^2+\cdots+c_{1,2n}e^{2n}+d_{11}f^1+d_{12}f^2+\cdots+d_{1,2n}f^{2n},\cdots,\\ c_{2k,1}e^1+c_{2k,2}e^2+\cdots+c_{2k,2n}e^{2n}+d_{2k,1}f^1+d_{2k,2}f^2+\cdots+d_{2k,2n}f^{2n}\}$$

且满足

(1) 秩 $A_{n+k}=$ 秩 $B_{n+k}=n+k$;

(2) $\alpha_i H\alpha_j^*=0, 1\le i,j\le 2n$;

其中

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,2n}\\ & \cdots & \cdots & \\ a_{n-k,1} & a_{n-k,2} & \cdots & a_{n-k,2n}\\ c_{11} & c_{12} & \cdots & c_{1,2n}\\ & \cdots & \cdots & \\ c_{2k,1} & c_{2k,2} & \cdots & c_{2k,2n}\end{pmatrix},\quad B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,2n}\\ & \cdots & \cdots & \\ b_{n-k,1} & b_{n-k,2} & \cdots & b_{n-k,2n}\\ d_{11} & d_{12} & \cdots & d_{1,2n}\\ & \cdots & \cdots & \\ d_{2k,1} & d_{2k,2} & \cdots & d_{2k,2n}\end{pmatrix}.$$

$$\alpha_i=\begin{cases}(a_{i1},\cdots,a_{i,2n},0,\cdots,0), & 1\le i\le n-1,\\ (0,\cdots,0,b_{i1},\cdots,b_{i,2n}), & n\le i\le 2n-2,\\ (c_{i1},\cdots,c_{i,2n},d_{i1},\cdots,d_{i,2n}), & 2n-1\le i\le 2n.\end{cases}$$

证明 类似于定理 3.4 的证明.

推论 3.9 **L** 是 **S** 的 k- 级完全 Langrangian 子流形 $\Longleftrightarrow \exists a_{ij},b_{ij},c_{lj},d_{lj}\in C, i=1,2,\cdots,n-k, j=1,2,\cdots,2n, l=1,2,\cdots,2k$, 使得

$$= \{\mathbf{f}\in \mathbf{S} \mid \exists s_i,t_i,u_j\in C, i=1,2,\cdots,n-k, j=1,2,\cdots,2k,\\ (f(a),f^{[1]}(a),\cdots,f^{[2n-1]}(a))^T=A_{n+k}^T(s_1,s_2,\cdots,s_{n-k},u_1,u_2,\cdots,u_{2k})^T,\\ (f(b),f^{[1]}(b),\cdots,f^{[2n-1]}(b))^T=B_{n+k}^T(t_1,t_2,\cdots,t_{n-k},u_1,u_2,\cdots,u_{2k})^T\},$$

且满足

(1) 秩 $A_{n+k}=$ 秩 $B_{n+k}=n+k$;

(2) $\alpha_i H\alpha_j^*=0, 1\le i,j\le 2n$;

其中

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,2n} \\ & \cdots & \cdots & \\ a_{n-k,1} & a_{n-k,2} & \cdots & a_{n-k,2n} \\ c_{11} & c_{12} & \cdots & c_{1,2n} \\ & \cdots & \cdots & \\ c_{2k,1} & c_{2k,2} & \cdots & c_{2k,2n} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1,2n} \\ & \cdots & \cdots & \\ b_{n-k,1} & b_{n-k,2} & \cdots & b_{n-k,2n} \\ d_{11} & d_{12} & \cdots & d_{1,2n} \\ & \cdots & \cdots & \\ d_{2k,1} & d_{2k,2} & \cdots & d_{2k,2n} \end{pmatrix}.$$

$$\alpha_i=\begin{cases} (a_{i1},\cdots,a_{i,2n},0,\cdots,0), & 1\le i\le n-1, \\ (0,\cdots,0,b_{i1},\cdots,b_{i,2n}), & n\le i\le 2n-2, \\ (c_{i1},\cdots,c_{i,2n},d_{i1},\cdots,d_{i,2n}), & 2n-1\le i\le 2n. \end{cases}$$

定理 3.10 **L** 是 **S** 的 n- 级完全 Lagrangian 子流形 $\Longleftrightarrow \exists c_{kj}, d_{kj}\in C, j=1,2,\cdots,2n,\quad k=1,2,\cdots,2n$, 使得

$$\mathbf{L}=\mathrm{span}\{c_{11}e^1+c_{12}e^2+\cdots+c_{1,2n}e^{2n}+d_{11}f^1+d_{12}f^2+\cdots+d_{1,2n}f^{2n},\cdots, c_{2n,1}e^1+c_{2n,2}e^2+\cdots+c_{2n,2n}e^{2n}+d_{2n,1}f^1+d_{2n,2}f^2+\cdots+d_{2n,2n}f^{2n}\}$$

且满足

(1) 秩 $A_{2n}=$ 秩 $B_{2n}=2n$;

(2) $\alpha_i H\alpha_j^*=0, 1\le i,j\le 2n$;

其中 $A_{2n}=(c_{sj})_{2n\times 2n}, B_{2n}=(d_{sj})_{2n\times 2n}, \alpha_i=(c_{i1},\cdots,c_{i,2n},d_{i1},\cdots,d_{i,2n}), i=1,2,\cdots,2n$.

推论 3.11 **L** 是 **S** 的 n- 级完全 Langrangian 子流形 $\Longleftrightarrow \exists c_{kj}, d_{kj}\in C, j=1,2,\cdots,2n,\quad k=1,2,\cdots,2n$, 使得

$$\mathbf{L}=\{\mathbf{f}\in\mathbf{S}|(A_{2n}^T)^{-1}(f(a),f^{[1]}(a),\cdots,f^{[2n-1]}(a))^T =(B_{2n}^T)^{-1}(f(b),f^{[1]}(b),\cdots,f^{[2n-1]}(b))^T\}.$$

且满足

(1) 秩 $A_{2n}=$ 秩 $B_{2n}=2n$;

(2) $\alpha_i H\alpha_j^*=0, 1\le i,j\le 2n$;

其中 $A_{2n}=(c_{sj})_{2n\times 2n}, B_{2n}=(d_{sj})_{2n\times 2n}, \alpha_i=(c_{i1},\cdots,c_{i,2n},d_{i1},\cdots,d_{i,2n}), i=1,2,\cdots,2n$.

例 2 $\mathbf{L}=\mathrm{span}\{e^1+f^1,e^2+f^2,\cdots,e^{2n}+f^{2n}\}=\{\mathbf{f}\in\mathbf{S}|f^{[k]}(a)=f^{[k]}(b),k=0,1,\cdots,2n-1\}$ 是 **S** 的一个 n- 级完全 Lagrangian 子流形.

注 从以上讨论可知，S 的完全 Lagrangian 子流形共分为 $n+1$ 类，都可以用边条件来描述. 但是，只有 0- 级完全 Lagrangian 子流形的边条件是分离的，其它类的边条件都是耦合的.

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Complex J-Symplectic Geometry Characterization for J-Symmetric Extensions of J-Symmetric Differential Operators

Abstrat We give complex J-symplectic geometry characterizations for J-symmetric extensions of J-symmetric ordinary differential operators.
Keywords Complex J-symplectic geometry, J-Lagrangian submanifold, J-Symmetric ordinary differential operator, J-Symmetric extension
MR(1991)Subject Classification 34B05,34L05,47B05,58F05
Chinese Library Classification O175.3

1 INTRODUCTION

The study of boundary value problem involving linear differential equation is becoming a well-established area of analysis. Applying the extension theroy of symmetric operators to concrete differential operators, a general characterization of self-adjoint extension of symmetric differential operators is established. For details of some of this work we refer to W. N. Everitt and Li marKus [1], Z. J. Cao [10], J. Sun [11], Z. J. Shang and R. Y. Zhu [12], S. Z. Fu [13], W. Y. Wang, J. Sun and Z. M. Zheng [14]. To study a similar problem in the case of J-symmetric differential expressions, for details of some of this work we refer to D. Race [2], Z. J. Shang [3], A. Galindo [4], I. Knowle [5,6], N. A. Zhikher [7].

[1] is concerned wth complex symplectic geometry with applications to ordinary differential operators. In [1], complex symplectic spaces and their lLagrangian subspaces are estaablished. And an appendix presents a related new result on the theory of self-adjoint operators in Hilbert spaces and this provides and important application of the principal theroems. In [1], comples symplectic geometry complete characterizations of self-adjoint extensions of symmetric operators are given.

[14] deal with complex J-symplectic geometry with application to ordinary differential operators and give complex J-symplectic geometry characterzations for J-selfadjoint extensions of J-symmetric operators.

In this paper, we give complex J-symplectic geometry characterizations for J-symmetric extensions of J-symmetric differential operatiors.

1 J-Symmetric extensons of J-Symmetric differential Operators

DEFINITION 1[14] A complex J-symplectic space S is a complex linear space, with a prescribed J-symplectic form $[:]$, namely, a be-linear form

(1) $u, v \to [u : v]$, $S \times S \to C$, so $[c_1 u + c_2 w : v] = c_1[u : v] + c_2[w, v]$,
where is skew-symmetirc,

(2) $[u : v] = -[v : u]$, so $[u : c_1 v + c_2 w] = c_1[u : v] + c_2[u : w]$,
and which is also non-degenerate,

(3) $[u : S] = 0$ imples $u = 0$,
for all vectors $u, v, w \in S$ and complex scalars $c_1, c_2 \in C$.

DEFINITION 2[14] A linear submanifold L in the complex J-symplectic space S is called J-Lagrangian in case $[L : L] = 0$, that is' $[u : v] = 0$ for all vectors $u, v \in L$.

Following we apply the complex J-symplectic genmetry to J-symmetric extensions of J-symmetric differential operators.

We consider the J-symmetric ordinary differential expression

$$\tau(y) = \sum_{k=0}^{n} (-1)^{n-k} (p_k(x) y^{(n-k)})^{(n-k)}$$

on the non-degenerate interval $I \subset R$, with endpoint $-\infty \leq a, b \leq +\infty$, where the functions $p_0^{-1}, p_1, \cdots, p_n$ are complex-valued, measurable over I and Lebesgue integrable on all compact subset of I and p_0 is non-vanishing.

We define the maximal opertor T_1 generated by τ in $L^2(I)$ as follows:

$$D(T_1) = \{y \in L^2(I) : y^{[k]} \in AC_{loc}(I) \ for \ 0 \le k \le 2n-1 \ and \ \tau(y) \in L^2(I)\},$$

$$T_1(y) = \tau(y) \quad for \quad y \in D(T_1),$$

where $y^{[k]}$ is the quasi-derivatives of $y(0 \le k \le 2n-1)$.

We then define the minimal operator T_0 as follows:

$$D(T_0) = \{y \in D(T_1) : [y : D(T_1)] = 0\},$$

$$T_0(y) = T_1(y) = \tau(y) \quad for \quad y \in D(T_0).$$

Here the skew-symmetric form [:] on $D(T_1)$ is given by

$$[y : z] = \langle z, JT_1(y)\rangle - \langle T_1(z), Jy\rangle \quad for \quad y, z \in D(T_1),$$

where

$$\langle y, z\rangle = \int_a^b y\bar{z}dx.$$

It is known [2,3] that $T_0 \subseteq T_1$ on $D(T_0) \subseteq D(T_1) \subset L^2(I)$ satisfy:

(1) $D(T_0)$ is dense in $L^2(I)$, so also $D(T_1)$ is dense in $L^2(I)$,

(2) $JT^*J = T_1$ and $JT_1^*J = T_0$,

so both T_0 and T_1 are closed operators and T_0 is J-symmetirc.

We now define the endpoint space $\mathbf{S}$, for M on I, as the quotient on iedntification vector space

$$\mathbf{S} = D(T_1)/D(T_0),$$

so there is a natural projection map

$$\Psi : D(T_1) \to \mathbf{S}, \quad f \to \mathbf{f} = \{f + D(T_0)\},$$

for $\mathbf{f} \in \mathbf{S}$,$f \in D(T_1)$.

We define the J-symplectic form [:] in $\mathbf{S}$

$$[\mathbf{f} : \mathbf{g}] = [f : g] \quad for \quad f, g \in D(T_1).$$

LEMMA 1[14]$\mathbf{S} = D(T_1)/D(T_0)$ is a complex J-symplectiv space

LEMMA 2 Suppose that T is a J-symmetric extension of T_0. Then $\mathbf{L} = D(T)/D(T_0)$ is a J-Lagrangian subspace of $\mathbf{S}$.

PROOF.Since $\mathbf{L} = D(T)/D(T_0)$ is the image of the linear manifold $D(T) \subseteq D(T_1)$ under the natural projection map

$$\Psi : D(T_1) \to \mathbf{S}, f \to \mathbf{f} = \{f + D(T_0)\}, \quad \text{for} \quad f \in D(T_1),$$

so $\mathbf{L}$ is a linear subspace of $\mathbf{S}$.

Since T is a J-symmetric extension of T_0, so we have

$$\begin{aligned}[\mathbf{f} : \mathbf{g}] = [f : g] &= \langle g, JT_1 f\rangle - \langle T_1 g, Jf\rangle \\ &= \langle g, JTf\rangle - \langle Tg, Jf\rangle \\ &= 0,\end{aligned}$$

for any $\mathbf{f}, \mathbf{g} \in \mathbf{L}$ where $f, g \in D(T)$. Hence $\mathbf{L} = D(T)/D(T_0)$ is a J-Lagrangian subspace of $\mathbf{S}$.

LEMMA 3.Suppose that $\mathbf{L}$ is a J-Lagrangian subspace of $\mathbf{S}$. Then the operator T as the restriction of T_1 to the domain

$$D(T) = \Psi^{-1}\mathbf{L} = \{f \in D(T_1) | \mathbf{f} \in \mathbf{L}\}$$

is a J-symmtric extension of T_0.

PROOFClearly $D(T)$ is a linear submanifold of $H = L^2(I)$ and

$$D(T_0) \subseteq D(T) \subseteq D(T_1).$$

Since $\mathbf{L}$ is a J-Lagrangian subspace of $\mathbf{S}$, so we have

$$[\mathbf{f} : \mathbf{g}] = \langle g, JTf\rangle - \langle Tg, Jf\rangle = 0,$$

for all $g \in D(T)$, whenever $f \in D(T)$. Thus we see that T ia a J-symmetric extension of T_0.

THEOREM 4.Suppose that T_0, T_1 and S are defined above. Then

(1) there exists a J-symmetric extension T of T_0 if and only if there exists a J-Lagrangian subspace $\mathbf{L}$ of $\mathbf{S}$;

(2) There exists a natural bi-unique correspondence between to set $\{T\}$ of all J-symmetric extensions T of T_0 and the set $\{\mathbf{L}\}$ of all J-Lagrangian subspaces $\mathbf{L}$ of $\mathbf{S}$.

PROOF(1) Using Lemma 2 and Lemma 3.

(2) The map induced by Ψ is

$\Phi : \{T\} \to \{\mathbf{L}\}$, defined by $T \to \mathbf{L} = D(T)/D(T_0)$.

Clearly Φ is surjective.

Following we show that Φ is jinjective. Take two different J-symmetric extensions T_α and T_β of T_0, with corresponding domains $D(T_\alpha)$ and $D(T_\beta)$. Let

$$\begin{aligned}\mathbf{L}_\alpha &= \Phi T_\alpha = D(T_\alpha)/D(T_0), \\ \mathbf{L}_\beta &= \Phi T_\beta = D(T_\beta)/D(T_0),\end{aligned}$$

By $T_\alpha \neq T_\beta$, we obtain $D(T_\alpha) \neq D(T_\beta)$. So there exists an element $u \in D(T_\alpha)$ but $u \notin D(T_\beta)$. In particular, $u \notin D(T_0)$. Then $\mathbf{u} = \{u + D(T_0)\}$ satisfies

$\mathbf{u} \in \mathbf{L}_\alpha$ but $\mathbf{u} \notin \mathbf{L}_\beta$.

Therefore the map Φ is injective. The theorem is completed.

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Poincaré Inequalities In Weighted Sobolev Spaces

Abstract — In this paper, we discuss the weighted Poincaré inequalities in weighted Sobolev spaces $W^{m,p}(\Omega; w, v_\alpha)$ and give some necessary and sufficient conditions for them to hold.

Key Words — Weighted sobolev spaces, Poincaré inequalities, Embeddings.

1 INTRODUCTION

The importance of the Poincaré inequalities in the theory of differential equations is well-known, and much effort has been devoted to the study of those inequalities, including the provision of necessary and sufficient condition for them to hold. For details of some of this work we refer to Amick[1], Edmunds and Evans[2,3], Hurri[4], Kufner and Opic[6], Edmunds and Opic[7,8,9]. [4] is concerned with weighted Poincaré inequalities. [7] is concerned with Poincaré inequalities in weighted Sobolev spaces $W^{1,p}(\Omega; w, v)$, necessary and sufficient conditions for the Poincaré inequalities hold if, and only if, the ball measure of non-compactness of the natural embedding of $W^{1,p}(\Omega; w, v)$ in $L^p(\Omega; w)$ is less than 1. [8,9] is concerned with the Poincaré inequalities in abstract Sobolev spaces.

In this paper we deal with the weighted Poincaré inequalities in weighted Sobolev spaces $W^{m,p}(\Omega; w, v_\alpha)$, give necessary and sufficient conditions for the weighted Poincaré inequalities to hold, that is, the Poincaré inequalities hold if, and only if, the ball measure of non-compactness of the natural embedding of $W^{m,p}(\Omega; w, v_\alpha)$ in $L^p(\Omega; w)$ is less than 1, and also give other forms of sufficient conditions for the weighted Poincaré inequalities to hold.

2 PRELIMINARIES

In this paper we shall consider real function space, but trivial modifications show that the results still hold for complex space.

Let Ω be a domain in R^n. By $W(\Omega)$ we denote the set of weight functions on Ω, that is the set of all measurable functions on Ω which are positive and finite almost everywhere. For $w \in W(\Omega)$, the weighted Lebesgue space $L^p(\Omega; w), 1 \leq p < \infty$, is the set of all measurable functions $u(x)$ on Ω with a finite norm

$$\|u(x)\|_{p,\Omega,w} = \left(\int_\Omega |u(x)|^p w(x) dx\right)^{\frac{1}{p}}. \tag{2.1}$$

Obviously the space $L(\Omega; w)$ with the norm (2.1) is complete.

Given any multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of non-negative integers α_i, we shall write

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \quad and \quad D^\alpha = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

Suppose $w, v_\alpha \in W(\Omega)$, the weighted Sobolev space $W^{m,p}(\Omega; w, v_\alpha), 0 \leq m < \infty$, consists of all those function $u(x)$ such that $D^\alpha u(x)$ exist on Ω for all α with $|\alpha| = m$ and this space is equipped with the norm

$$\|u(x)\|_{m,p,\Omega,w,v_\alpha} = \left(\|u(x)\|^p_{p,\Omega,w} + \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha}\right)^{\frac{1}{p}}. \tag{2.2}$$

In all this derivatives $D^\alpha u(x)$ are naturally assumed to be taken in the distributional sense.

LEMMA 2.1. (1) Let

$$w^{-\frac{1}{p}}, v_\alpha^{-\frac{1}{p}} \in L^q_{loc}(\Omega)$$

(q is the conjugate index of p). Then $W^{m,p}(\Omega; w, v_\alpha)$ is a Banach space.

(2) Let $w, v_\alpha \in L^1_{loc}(\Omega)$. Then $C_0^\infty(\Omega) \subset W^{m,p}(\Omega; w, v_\alpha)$.

(3) Let $w^{-\frac{1}{p}}, v_\alpha^{-\frac{1}{p}} \in L^q_{loc}(\Omega), w, v_\alpha \in L^1_{loc}(\Omega)$. Then $W_0^{m,p}(\Omega; w, v_\alpha)$ is a Banach space, where $W_0^{m,p}(\Omega; w, v_\alpha)$ is the closure of the set C_0^∞ with respect to the norm (2.2).

The proof is routine.

Throughout this paper we assume that $w^{-\frac{1}{p}}, v_\alpha^{-\frac{1}{p}} \in L^q_{loc}(\Omega)$ and $w, v_\alpha \in L^1_{loc}(\Omega)$.

Given two Banach space X, Y, we write $X \hookrightarrow Y$ or $X \hookrightarrow\hookrightarrow Y$ if $X \subset Y$ and the natural injection of X into Y is continuous or compact, respectively.

3 THE WEIGHTED POINCARÉ INEQUALITY

Suppose $1 \le p < \infty$ and w, $v_\alpha \in W(\Omega)$. We say that (Ω, w, v_α) supports the weighted Poincaré inequality if there is a positive constant K_1 such that

$$\int_\Omega |u(x)|^p w(x)dx \le K_1 \left[\left| \int_\Omega u(x)w(x)dx \right|^p + \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha} \right]$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

LEMMA 3.1. Let $w \in W(\Omega)$ and

$$\int_\Omega w(x)dx < \infty.$$

Then the weighted average $u_{\Omega,w}$ of a function u over Ω,

$$u_{\Omega,w} = \left(\int_\Omega w(x)dx \right)^{-1} \left(\int_\Omega u(x)w(x)dx \right),$$

is finite for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

PROOF. Using Hölder's inequality, we obtain

$$\begin{aligned} |u_{\Omega,w}| &= \left(\int_\Omega w(x)dx \right)^{-1} \left| \int_\Omega u(x)w(x)dx \right| \le \left(\int_\Omega w(x)dx \right)^{-1} \int_\Omega |u(x)||w(x)|^{\frac{1}{p}} |w(x)|^{\frac{1}{q}} dx \\ &\le \left(\int_\Omega w(x)dx \right)^{-\frac{1}{p}} \|u(x)\|_{p,\Omega,w}. \end{aligned}$$

THEOREM 3.2. Let w,$v_\alpha \in W(\Omega), w \in L^1(\Omega)$. Then the following conditions are equivalent:

(1) (Ω, w, v_α) supports the weighted Poincaré inequality.

(2) there is a positive constant K_2 such that

$$\int_\Omega |u - u_{\Omega,w}|^p w dx \le K_2 \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

(3) there is a positive constant K_3 such that

$$\inf_{c \in R} \|u(x) - c\|_{p,\Omega,w} \le K_3 \left(\sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha} \right)^{\frac{1}{p}}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

PROOF. (1) $\Rightarrow$ (2): Let $u(x) \in W^{m,p}(\Omega; w, v_\alpha)$. Then the function $u(x) - u_{\Omega,w} \in W^{m,p}(\Omega; w, v_\alpha)$ and

$$\int_\Omega (u(x) - u_{\Omega,w}) w dx = 0.$$

By (1), we have

$$\int_\Omega |u(x) - u_{\Omega,w}|^p w dx \le K_1 \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha}.$$

(2) $\Rightarrow$ (3): Let $u(x) \in W^{m,p}(\Omega; w, v_\alpha)$.

$$\inf_{c\in R} \|u(x) - c\|_{p,\Omega,w} \le \|u(x) - u_{\Omega,w}\|_{p,\Omega,w} \le K_2^{\frac{1}{p}} \left(\sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha} \right)^{\frac{1}{p}}.$$

(3) $\Rightarrow$ (2): For $u(x) \in W^{m,p}(\Omega; w, v_\alpha)$ and $c \in R$ we have

$$\|u(x) - u_{\Omega,w}\|_{p,\Omega,w} \le \|u(x) - c\|_{p,\Omega,w} + \|c - u_{\Omega,w}\|_{p,\Omega,w}. \tag{3.1}$$

Moreover, using Hölder's inequality, we obtain

$$\begin{aligned}
\|u_{\Omega,w} - c\|_{p,\Omega,w} &= \left[\int_\Omega |u_{\Omega,w} - c|^p w dx\right]^{\frac{1}{p}} \\
&= \left[\int_\Omega \left| \left(\int_\Omega w(y)dy\right)^{-1} \left(\int_\Omega u(y)w(y)dy\right) - c\right|^p w(x)dx\right]^{\frac{1}{p}} \\
&= \left(\int_\Omega w(y)dy\right)^{-1} \left[\int_\Omega \left|\int_\Omega (u(y) - c)w(y)dy\right|^p w(x)dx\right]^{\frac{1}{p}} \\
&= \left(\int_\Omega w(y)dy\right)^{\frac{1}{p}-1} \left|\int_\Omega (u(y) - c)w(y)dy\right| \\
&\le \left(\int_\Omega w(y)dy\right)^{-\frac{1}{q}} \left(\int_\Omega |u(y) - c|^p w(y)dy\right)^{\frac{1}{p}} \left(\int_\Omega w(y)dy\right)^{\frac{1}{q}} \\
&= \|u - c\|_{p,\Omega,w}.
\end{aligned}$$

This inequality and (3.1) imply

$$\|u(x) - u_{\Omega,w}\|_{p,\Omega,w} \le 2 \inf_{c\in R} \|u(x) - c\|_{p,\Omega,w}. \tag{3.2}$$

Thus we have

$$\begin{aligned}
\int_\Omega |u(x) - u_{\Omega,w}|^p w dx &= \|u(x) - u_{\Omega,w}\|^p_{p,\Omega,w} \\
&\le \left(2 \inf_{c\in R} \|u(x) - c\|_{p,\Omega,w}\right)^p \\
&\le \left(2K_3 \left(\sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha}\right)^{\frac{1}{p}}\right)^p \\
&= (2K_3)^p \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha}
\end{aligned}$$

(2) $\Rightarrow$ (1): Let $u(x) \in W^{m,p}(\Omega; w, v_\alpha)$. Since

$$\|u(x)\|_{p,\Omega,w} \le \|u_{\Omega,w}\|_{p,\Omega,w} + \|u - u_{\Omega,w}\|_{p,\Omega,w}$$

and

$$\begin{aligned} \|u_{\Omega,w}\|_{p,\Omega,w} &= \left(\int_\Omega |u_{\Omega,w}|^p w dx\right)^{\frac{1}{p}} \\ &= |u_{\Omega,w}| \left(\int_\Omega w dx\right)^{\frac{1}{p}} \\ &= \left| \left(\int_\Omega w dx\right)^{-1} \int_\Omega u(x)w(x)dx \right| \left(\int_\Omega w dx\right)^{\frac{1}{p}} \\ &= \left(\int_\Omega w dx\right)^{-\frac{1}{q}} \left|\int_\Omega u(x)w(x)dx\right| \end{aligned}$$

We have

$$\|u(x)\|_{p,\Omega,w} \leq \left(\int_\Omega w dx\right)^{-\frac{1}{q}} \left|\int_\Omega u(x)w(x)dx\right| + \|u(x) - u_{\Omega,w}\|_{p,\Omega,w}. \tag{3.3}$$

By (3.3), together (2) yields

$$\begin{aligned} \int_\Omega |u|^p w dx &= \|u\|^p_{p,\Omega,w} \\ &\leq \left[\left(\int_\Omega w dx\right)^{\frac{1}{q}} \left|\int_\Omega u(x)w(x)dx\right| + \|u(x) - u_{\Omega,w}\|_{p,\Omega,w}\right]^p \\ &\leq 2^{p-1}\left[\left(\int_\Omega w dx\right)^{\frac{p}{q}} \left|\int_\Omega u(x)w(x)dx\right|^p + \|u - u_{\Omega,w}\|^p_{p,\Omega,w}\right] \\ &\leq 2^{p-1}\left[\left(\int_\Omega w dx\right)^{\frac{p}{q}} \left|\int_\Omega u(x)w(x)dx\right|^p + K_2 \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha}\right] \\ &\leq K_1 \left(\left|\int_\Omega u(x)w(x)dx\right|^p dx + \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha}\right). \end{aligned}$$

where

$$K_1 = 2^{p-1} \max\left\{K_2, \left(\int_\Omega w dx\right)^{\frac{p}{q}}\right\}.$$

The proof is completed.

By $W_c(\Omega)$ we denote a special subclass of $W(\Omega)$, that is the set of all $w \in W(\Omega)$ which are bounded from above and from below by positive constants on each compact subset Q in Ω. Consequently, $w \in L^1_{loc}(\Omega)$ and $w^{-\frac{1}{p}} \in L^q_{loc}(\Omega)$ if $w \in W(\Omega)$ (see, [7] p.82).

For an arbitrary domain $\Omega \subset R^n$ we can write

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k \tag{3.4}$$

where $\Omega_k \in C^{0,1}$ (that is, Ω_k is a bounded domain whose boundary can be locally described by functions satisfying a Lipschitz condition) (see, [5] p.21 and [7] p.83) and $\Omega_k \subset \overline{\Omega}_k \subset \Omega_{k+1} \subset \Omega$ for each $k \in N$.

Put $\Omega^k = \Omega \backslash \Omega_k$ and define

$$A_k = \sup_{\|u\|_{m,p,\Omega,w,v_\alpha} \leq 1} \|u\|_{p,\Omega^k,w}. \tag{3.5}$$

By (3.5) and Remark 3.10. of [7] we have

$$0 \leq A_{k+1} \leq A_k \leq 1, \quad \lim_{k\to\infty} A_k = A \in [0,1], \quad A = \bar{\beta}(I).$$

where $\tilde{\beta}(I)$ be the ball measure of non-compactness of the embedding

$$I : W^{m,p}(\Omega; w, v_\alpha) \to L^p(\Omega; w).$$

THEOREM 3.3. Suppose $1 \le p < \infty$ and w, $v_\alpha \in W_c(\Omega)$.Let F be a functional on $W^{m,p}(\Omega; w, v_\alpha)$ with the following properties:

$(c_1)F$ is continuous on $W^{m,p}(\Omega; w, v_\alpha)$,

$(c_2)F(\lambda u) = \lambda F(u)$ for all $\lambda > 0$ and all $u \in W^{m,p}(\Omega; w, v_\alpha)$,

and

(c_3) if $u \in P_{m-1} \cap W^{m,p}(\Omega; w, v_\alpha)(P_{m-1}$ is the set of all polynomials on R^n of degree less than m) and $F(u) = 0$, then $u = 0$.

Let $A < 1$. Then there is a constant K such that

$$\int_\Omega |u|^p w dx \le K \left[|F(u)|^p + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{p,\Omega,v_\alpha} \right] \tag{3.6}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

PROOF. Let $\alpha \in (A, 1)$. Then there exists $n_0 \in N$ such that

$$A_n \le \alpha \;\; for\ all \;\; n \ge n_0. \tag{3.7}$$

Fix $n \in N, n \ge n_0$, then by (3.5) and (3.7) we have

$$\int_{\Omega^n} |u|^p w dx \le \alpha^p \|u\|^p_{m,p,\Omega,w,v_\alpha},$$

and

$$\int_{\Omega^n} |u|^p w dx \le \frac{\alpha^p}{1-\alpha^p} \left(\|u(x)\|^p_{p,\Omega_n,w} + \sum_{|\alpha|=m} \|D^\alpha u(x)\|^p_{p,\Omega,v_\alpha} \right) \tag{3.8}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

Suppose that Theorem 3.3 is false. Then there is a sequence $\{u_j\} \subset W^{m,p}(\Omega; w, v_\alpha)$ such that

$$\int_\Omega |u_j|^p w dx = 1 \;\; for\ all \;\; j \in N, \tag{3.9}$$

$$F(u_j) \to 0 \;\; as \;\; j \to \infty \tag{3.10}$$

and

$$\sum_{|\alpha|=m} \|D^\alpha u_j\|^p_{p,\Omega,v_\alpha} \to 0 \;\; as \;\; j \to \infty. \tag{3.11}$$

Since $w, v_\alpha \in W_c(\Omega)$ and $\Omega_n \in C^{0,1}$, we have

$$W^{m,p}(\Omega_n; w, v_\alpha) \hookrightarrow\hookrightarrow L^p(\Omega_n; w)$$

By (3.9) and (3.11), the sequence $\{u_j\}$ is bounded in $W^{m,p}(\Omega_n; w, v_\alpha)$, consequently there is a subsequence $\{u_{j(k)}\}$ which is a Cauchy sequence in $L^p(\Omega_n; w)$. This, (3.8) and (3.11) together imply that $\{u_{j(k)}\}$ is also a Cauchy subsequence in $L^p(\Omega^n; w)$. Hence, there exists $u \in L^p(\Omega; w)$ such that

$$u_{j(k)} \to u \;\; in \;\; L^p(\Omega; w) \;\; as \;\; k \to \infty \tag{3.12}$$

By (3.12) and (3.11), we have

$$u_{j(k)} \to u \;\; in \;\; W^{m,p}(\Omega; w, v_\alpha) \;\; as \;\; k \to \infty.$$

According to $(c_1), F(u_{j(k)}) \to F(u)$ $(k \to \infty)$. By (3.10), we have $F(u) = 0$.

Let $\phi \in C_0^\infty(\Omega)$. Then by (3.12) and (3.11),

$$\int_\Omega \phi D^\alpha u dx = (-1)^m \int_\Omega u D^\alpha \phi dx = (-1)^m \lim_{k\to\infty} \int_\Omega u_{j(k)} D^\alpha \phi dx = \lim_{k\to\infty} \int_\Omega \phi D^\alpha u_{j(k)} dx = 0$$

that is, $D^\alpha u = 0$ in $\Omega(|\alpha| = m)$, thus $u \in P_{m-1}$. By (c_3), we have $u = 0$, which contradicts (3.9) and (3.12). The theorem is proved.

COROLLARY 3.4. Suppose $1 \le p < \infty$ and w, $v_\alpha \in W_c(\Omega)$.Let $A < 1$ and the following condition hold:

(c_4) if $u \in P_{m-1} \cap W^{m,p}(\Omega; w, v_\alpha)$ and $\int_\Omega uwdx = 0$, then $u = 0$.

Then (Ω, w, v_α) supports the weighted Poincaré inequality.

PROOF. Using Theorem 3.3. with $F(u) = \int_\Omega uwdx$.

COROLLARY 3.5. Suppose $1 \le p < \infty$ and w, $v_\alpha \in W_c(\Omega)$.Let $P_{m-1} \cap W^{m,p}(\Omega; w, v_\alpha) = \{0\}$, $A < 1$. Then there is a constant K such that

$$\int_\Omega |u|^p wdx \le K \sum_{|\alpha|=m} \|D^\alpha u\|^p_{p,\Omega,v_\alpha} \tag{3.13}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

PROOF. Using Theorem 3.3. with $F(u) = 0$.

THEOREM 3.6. Suppose $1 \le p < \infty$ and w, $v_\alpha \in W_c(\Omega)$,let $F \in W^{m,p}(\Omega; w, v_\alpha)^*$ and $(c_1), (c_2), (c_3)$ hold.If there exists a constant K such that the inequality (3.6) holds for all $u \in W^{m,p}(\Omega; w, v_\alpha)$, then $A < 1$.

PROOF. By the definition of A_j there is a function $u_j \in W^{m,p}(\Omega; w, v_\alpha)$, with $\|u_j\|_{m,p,\Omega,w,v_\alpha} \le 1$, such that

$$A_j - \frac{1}{j} \le \|u_j\|_{p,\Omega^j,w} \le A_j \quad (j \in N). \tag{3.14}$$

Assume that $\lim_{j\to\infty} A_j = A = 1$. Then (3.14) yields

$$\lim_{j\to\infty} \|u_j\|_{p,\Omega^j,w} = 1. \tag{3.15}$$

Since

$$\|u_j\|_{p,\Omega^j,w} \le \|u_j\|_{p,\Omega,w} \le \|u_j\|_{m,p,\Omega,w,v_\alpha} \le 1$$

the equality (3.15) implies

$$\|u_j\|_{p,\Omega,w} \to 1 \quad as \quad j \to \infty, \tag{3.16}$$

$$\|u_j\|_{m,p,\Omega,w,v_\alpha} \to 1 \quad as \quad j \to \infty, \tag{3.17}$$

and consequently

$$\sum_{|\alpha|=m} \|D^\alpha u_j\|^p_{p,\Omega,v_\alpha} \to 0 \quad as \quad j \to \infty, \tag{3.18}$$

$$\|u_j\|_{p,\Omega_j,w} \to 0 \quad as \quad j \to \infty, \tag{3.19}$$

By (3.17), the sequence $\{u_j\}$ is bounded in $W^{m,p}(\Omega; w, v_\alpha)$, and, consequently, the sequence of real numbers $\{F(u_j)\}$ is also bounded. Thus there is a Cauchy subsequence $\{F(u_{j(k)})\}$ of $\{F(u_j)\}$.

Using (3.6), we have

$$\begin{aligned}
&\int_\Omega |u_{j(k)} - u_{j(l)}|^p wdx \\
&\quad \le K \left[|F(u_{j(k)} - u_{j(l)})|^p + \sum_{|\alpha|=m} \|D^\alpha (u_{j(k)} - u_{j(l)})\|^P_{p,\Omega,v_\alpha} \right] \\
&\quad \le K \left[|F(u_{j(k)} - u_{j(l)})|^p + 2^{p-1} \sum_{|\alpha|=m} \|D^\alpha (u_{j(k)})\|^P_{p,\Omega,v_\alpha} + 2^{p-1} \sum_{|\alpha|=m} \|D^\alpha (u_{j(l)})\|^P_{p,\Omega,v_\alpha} \right]
\end{aligned}$$

By (3.18), the sequence $\{u_{j(k)}\}$ is the Cauchy sequence in $L^p(\Omega; w)$. Therefore there exists $u \in L^p(\Omega; w)$ such that

$$u_{j(k)} \to u \quad in \quad L^p(\Omega; w) \quad as \quad j \to \infty. \tag{3.20}$$

Using (3.19), we have

$$0 \leq \|u\|_{p,\Omega,w} \leq \|u - u_{j(k)}\|_{p,\Omega,w} + \|u_{j(k)}\|_{p,\Omega_{j(k)},w} \to 0.$$

Hence $u = 0$, which contradicts (3.16) and (3.20). The theorem is proved.

COROLLARY 3.7. Suppose $1 \leq p < \infty$, $w, v_\alpha \in W_c(\Omega)$ and $w \in L^1(\Omega)$ and let condition (c_4) holds. If (Ω, w, v_α) supports the weighted Poincaré inequality, then $A < 1$.

COROLLARY 3.8. Suppose $1 \leq p < \infty$, $w, v_\alpha \in W_c(\Omega)$ and let $P_{m-1} \cap W^{m,p}(\Omega; w, v_\alpha) = \{0\}$. If there exist a constant K such that the inequality (3.7) hold for all $u \in W^{m,p}(\Omega; w, v_\alpha)$, then $A < 1$.

From Theorem 3.3 and 3.6 we have

THEOREM 3.9. Suppose $1 \leq p < \infty$ w, $v_\alpha \in W_c(\omega)$, let $F \in W^{m,p}(\Omega; w, v_\alpha)^*$ and conditions $(c_1), (c_2), (c_3)$ hold. Then there exist a constant K such that the inequality

$$\int_\Omega |u|^p w dx \leq K \left[|F(u)|^p + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{p,\Omega,v_\alpha} \right]$$

hold for all $u \in W^{m,p}(\Omega; w, v_\alpha)$ if, and only if, $A < 1$.

From Corollary 3.4 and 3.7 we have

THEOREM 3.10. Suppose $1 \leq p < \infty, w, v_\alpha \in W_c(\Omega), w \in L^1(\Omega)$ and let condition (c_4) holds. Then (Ω, w, v_α) supports the weighted Poincaré inequality if, and only if, $A < 1$.

From Corollary 3.5 and 3.8 we have

THEOREM 3.11. Suppose $1 \leq p < \infty$ and w, $v_\alpha \in W_c(\Omega)$.Let $P_{m-1} \cap W^{m,p}(\Omega; w, v_\alpha) = \{0\}$. Then there exists a constant K such that the inequality

$$\int_\Omega |u|^p w dx \leq K \sum_{|\alpha|=m} \|D^\alpha u\|^p_{p,\Omega,v_\alpha}$$

hold for all $u \in W^{m,p}(\Omega; w, v_\alpha)$ if, and only if, $A < 1$.

REMARK. Above necessary and sufficient conditions, we have got, for the weighted Poincaré inequalities to hold use the ball measure of non-compactness of the natural embedding of $W^{m,p}(\Omega; w, v_\alpha)$ in $L^p(\Omega; w)$. In the following, we try to discuss same problem by the compact embedding of $W^{m,p}(\Omega; w, v_\alpha)$ in $X(\Omega)$ or $L^p(\Omega, v)$.

THEOREM 3.12. Suppose $1 \leq p < \infty$ and w, $v_\alpha \in W_c(\Omega)$.Let $F \in W^{m,p}(\Omega; w, v_\alpha)^*$ and conditions $(c_1), (c_2), (c_3)$ hold. If one of the following conditions (c_5), (c_6) and (c_7) is satisfied,

$(c_5) W^{m,p}(\Omega; w, v_\alpha) \hookrightarrow\hookrightarrow X(\Omega)$ and there is a constant C such that

$$\|u\|_{p,\Omega,w} \leq C\|u\|_{X(\Omega)}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

$(c_6) W^{m,p}(\Omega; w, v_\alpha) \hookrightarrow\hookrightarrow X(\Omega)$ and there is a constant C such that

$$\|u\|_{m,p,\Omega,w,v_\alpha} \leq C\|u\|_{X(\Omega)}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

(c_7) There exists $v \in W_c(\Omega)$ such that

$$W^{m,p}(\Omega; w, v_\alpha) \hookrightarrow\hookrightarrow L^p(\Omega; v)$$

and $w \leq v$ on Ω, then there is a constant K such that

$$\int_\Omega |u|^p w dx \leq K \left[|F(u)|^p + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{p,\Omega,v_\alpha} \right]$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

PROOF. (c_5) is satisfied.
Suppose that Theorem 3.12 is false. Then there is a sequence $\{u_j\} \subset W^{m,p}(\Omega; w, v_\alpha)$ such that

$$\int_\Omega |u_j|^p w dx = 1 \quad for\ all \quad j \in N \tag{3.21}$$

$$F(u_j) \to 0 \quad as \quad j \to \infty \tag{3.22}$$

and

$$\sum_{|\alpha|=m} \|D^\alpha u_j\|^p_{p,\Omega,v_\alpha} \to 0 \quad as \quad j \to \infty \tag{3.23}$$

By (3.21) and (3.23), the sequence $\{u_j\}$ is bounded in $W^{m,p}(\Omega; w, v_\alpha)$. Since $W^{m,p}(\Omega; w, v_\alpha) \hookrightarrow\hookrightarrow X(\Omega)$, consequently there is a subsequence $\{u_{j(k)}\}$ which is a Cauchy sequence in $X(\Omega)$, that is, there exists $u \in X(\Omega)$ such that

$$u_{j(k)} \to u \quad in \quad X(\Omega) \quad as \quad k \to \infty \tag{3.24}$$

By $\|u_{j(k)}\|_{p,\Omega,w} \le C\|u_{j(k)}\|_{X(\Omega)}$, we have

$$u_{j(k)} \to u \quad in \quad L^p(\Omega; w) \quad as \quad k \to \infty \tag{3.25}$$

By (3.25) and (3.23), we have

$$u_{j(k)} \to u \quad in \quad W^{m,p}(\Omega; w, v_\alpha)^\cdot \quad as \quad k \to \infty.$$

According to $(c_1), F(u_{j(k)}) \to F(u)(k \to \infty)$. By (3.22), we have $F(u) = 0$.
Let $\phi \in C_0^\infty(\Omega)$. Then by (3.24) and (3.23),

$$\int_\Omega \phi D^\alpha u dx = (-1)^m \int_\Omega u D^\alpha \phi dx = (-1)^m \lim_{k\to\infty} \int_\Omega u_{j(k)} D^\alpha \phi dx = \lim_{k\to\infty} \int_\Omega \phi D^\alpha u_{j(k)} dx = 0$$

that is, $D^\alpha u = 0$ in $\Omega(|\alpha| = m)$, thus $u \in P_{m-1}$. By (c_3), we have $u = 0$, which contradicts (3.21) and (3.24). Similar discussion is available for cases (c_6) and (c_7) and the theorem is proved.

COROLLARY 3.13. Suppose $1 \le p < \infty, w, v_\alpha \in W_c(\Omega), w \in L^1(\Omega)$ and let (c_4) hold. If one of the conditions $(c_5), (c_6)$, and (c_7) is satisfied, then (Ω, w, v_α) supports the weighted Poincaré inequality.

COROLLARY 3.14. Suppose $1 \le p < \infty$ and w, $v_\alpha \in W_c(\Omega)$.Let $P_{m-1} \cap W^{m,p}(\Omega; w, v_\alpha) = \{0\}$. If one of the conditions $(c_5), (c_6)$, and (c_7) is satisfied, then there is a constant K such that

$$\int_\Omega |u|^p w dx \le K \sum_{|\alpha|=m} \|D^\alpha u\|^p_{p,\Omega,v_\alpha}$$

for all $u \in W^{m,p}(\Omega; w, v_\alpha)$.

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Weighted Poincaré Inequalities on One-Dimensional Unbounded Domains

Abstract—— In this paper, we discuss the weighted Poincaré inequalities on one-dimensional unbounded domains and give sufficient conditions for them to hold.
Key Words——Weighted sobolev spaces, Poincaré inequalities, Unbounded domains.

1 INTRODUCTION

The importance of the Poincaré inequalities in the theory of differential equations is well-known, and much effort has been devoted to the study of those inequalities, including the provision of necessary and sufficient condition for them to hold. For details of some of this work we refer to Amick[1], Edmunds and Evans[2,3], Hurri[4], Kufner and Opic[6], Edmunds and Opic[7,8,9], Elcrat and Maclean [10], Wang, Sun and Zheng [11]. [4] is concerned with weighted Poincaré inequalities. [7] is concerned with Poincaré inequalities in weighted Sobolev spaces $W^{1,p}(\Omega; w, v)$, necesary and sufficient conditions for the Poincaré inequality to hold are given. [8,9] show the Poincaré inequalities in abstract Sobolev spaces. [10] presents Poincaré inequalities in the weighted Sobolev spaces $W^{1,p}(\Omega; \phi, \phi)$, where Ω is one-dimensional unbounded domain. We discuss the weighted Poincaré ineqùalities of the weighted Sobolev spaces $W^{m,p}(\Omega; w, v_\alpha)$ in [11].

In the present paper, we deal with the weighted Poincaré inequalities in weighted Sobolev spaces $W^{m,p}(\Omega; x^\alpha, x^\beta)$ and $W^{m,p}(\Omega; w, w)$, where Ω is one-dimensional unbounded domain, and give sufficient conditions for the weighted Poincaré inequalities to hold.

2 PRELIMINARIES

Let Ω be a one-dimensional unbounded domain in R. Suppose $1 \le p < \infty$ and $0 \le m < \infty, \alpha, \beta \in R$. The weighted Sobolev space $W^{m,p}(\Omega; x^\alpha, x^\beta)$ consist of all those function $u(x)$ such that $u^{(m)}(x)$ exist on Ω and this space is equipped with the norm

$$\|u(x)\|_{m,p,\Omega,x^\alpha,x^\beta} = (\|u(x)\|^p_{p,\Omega,x^\alpha} + \|u^{(m)}(x)\|^p_{p,\Omega,x^\beta})^{\frac{1}{p}}$$

where

$$\|u(x)\|_{p,\Omega,x^\alpha} = \left(\int_\Omega |u(x)|^p x^\alpha dx\right)^{\frac{1}{p}}.$$

LEMMA2.1. Let $x^\alpha \in L^1(\Omega)$. Then the weighted average u_{Ω,x^α} of a function $u(x)$ over Ω,

$$u_{\Omega,x^\alpha} = \left(\int_\Omega x^\alpha dx\right)^{-1} \left(\int_\Omega u(x) x^\alpha dx\right),$$

is finite for all $u \in W^{m,p}(\Omega; x^\alpha, x^\beta)$.

The proof is given by [11].

LEMMA2.2. Let $1 \le p < \infty, 0 \le m < \infty, a > 0, \Omega = (a, \infty), \alpha, \beta \in R, x^\alpha \in L^1(\Omega)$. Then the following conditions are equivalent:

(1) there exists a positive constant K_1 such that

$$\int_\Omega |u(x)|^p x^\alpha dx \le K_1 \left(\left|\int_\Omega u(x) x^\alpha dx\right|^p + \|u^{(m)}(x)\|^p_{p,\Omega,x^\beta}\right);$$

(2) there exists a positive constant K_2 such that

$$\|u(x) - u_{\Omega,x^\alpha}\| \le K_2 \|u^{(m)}(x)\|_{p,\Omega,x^\beta};$$

(3) there exists a positive constant K_3 such that

$$\inf_{c \in R} \|u - c\|_{p,\Omega,x^\alpha} \le K_3 \|u^{(m)}(x)\|_{p,\Omega,x^\beta};$$

where $u(x) \in W^{m,p}(\Omega; x^\alpha, x^\beta)$.

PROOF. (1) ⇒ (2): Let $u(x) \in W^{m,p}(\Omega; x^\alpha, x^\beta)$. Then the function $u(x) - u_{\Omega,x^\alpha} \in W^{m,p}(\Omega; x^\alpha, x^\beta)$ and

$$\int_\Omega (u(x) - u_{\Omega,x^\alpha}) x^\alpha dx = 0.$$

By (1), we have

$$\|u(x) - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha} = \left(\int_\Omega |u(x) - u_{\Omega,x^\alpha}|^p x^\alpha dx\right)^{\frac{1}{p}} \leq K_1^{\frac{1}{p}} \|u^{(m)}(x)\|_{p,\Omega,x^\beta}.$$

(2) ⇒ (3): Let $u(x) \in W^{m,p}(\Omega; x^\alpha, x^\beta)$.

$$\inf_{c\in R} \|u(x) - c\|_{p,\Omega,x^\alpha} \leq \|u(x) - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha} \leq K_2 \|u^{(m)}(x)\|_{p,\Omega,x^\beta}.$$

(3) ⇒ (2): For $u(x) \in W^{m,p}(\Omega; x^\alpha, x^\beta)$ and $c \in R$ we have

$$\|u(x) - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha} \leq \|u(x) - c\|_{p,\Omega,x^\alpha} + \|c - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha}. \tag{2.1}$$

Moreover, using Hölder's inequality, we obtain

$$\begin{aligned}
\|u_{\Omega,x^\alpha} - c\|_{p,\Omega,x^\alpha} &= \left[\int_\Omega |u_{\Omega,x^\alpha} - c|^p x^\alpha dx\right]^{\frac{1}{p}} \\
&= \left[\int_\Omega \left|\left(\int_\Omega y^\alpha dy\right)^{-1} \left(\int_\Omega u(y) y^\alpha dy\right) - c\right|^p x^\alpha dx\right]^{\frac{1}{p}} \\
&= \left(\int_\Omega y^\alpha dy\right)^{-1} \left[\int_\Omega \left|\int_\Omega (u(y) - c) y^\alpha dy\right|^p x^\alpha dx\right]^{\frac{1}{p}} \\
&= \left(\int_\Omega y^\alpha dy\right)^{\frac{1}{p}-1} \left|\int_\Omega (u(y) - c) y^\alpha dy\right| \\
&\leq \left(\int_\Omega y^\alpha dy\right)^{-\frac{1}{q}} \left(\int_\Omega |(u(y) - c)|^p y^\alpha dy\right)^{\frac{1}{p}} \left(\int_\Omega y^\alpha dy\right)^{\frac{1}{q}}) \\
&= \|u - c\|_{p,\Omega,x^\alpha}
\end{aligned}$$

(q is the conjugate index of p). This inequality and (2.1) imply

$$\|u(x) - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha} \leq 2 \inf_{c\in R} \|u(x) - c\|_{p,\Omega,x^\alpha} \leq 2K_3 \|u^{(m)}\|_{p,\Omega,x^\alpha}. \tag{2.2}$$

(2) ⇒ (1): Let $u(x) \in W^{m,p}(\Omega; x^\alpha, x^\beta)$. Since

$$\|u(x)\|_{p,\Omega,x^\alpha} \leq \|u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha} + \|u - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha}.$$

and

$$\begin{aligned}
\|u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha} &= \left(\int_\Omega |u_{\Omega,x^\alpha}|^p x^\alpha dx\right)^{\frac{1}{p}} \\
&= |u_{\Omega,x^\alpha}| \left(\int_\Omega x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left|\left(\int_\Omega x^\alpha dx\right)^{-1} \int_\Omega u(x) x^\alpha dx\right| \left(\int_\Omega x^\alpha dx\right)^{\frac{1}{p}} \\
&= \left(\int_\Omega x^\alpha dx\right)^{-\frac{1}{q}} \left|\int_\Omega u(x) x^\alpha dx\right|
\end{aligned}$$

We have

$$\|u(x)\|_{p,\Omega,x^\alpha} \leq \left(\int_\Omega x^\alpha dx\right)^{-\frac{1}{q}} \left|\int_\Omega u(x) x^\alpha dx\right| + \|u(x) - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha}.$$

This inequality, together with (2), yields

$$\begin{aligned}
\int_\Omega |u|^p x^\alpha dx &= \|u\|^p_{p,\Omega,x^\alpha} \\
&\le \left[\left(\int_\Omega x^\alpha dx\right)^{\frac{1}{q}} \left|\int_\Omega u(x)x^\alpha dx\right| + \|u(x) - u_{\Omega,x^\alpha}\|_{p,\Omega,x^\alpha}\right]^p \\
&\le 2^{p-1}\left[\left(\int_\Omega x^\alpha dx\right)^{\frac{p}{q}} \left|\int_\Omega u(x)x^\alpha dx\right|^p + \|u - u_{\Omega,x^\alpha}\|^p_{p,\Omega,x^\alpha}\right] \\
&\le 2^{p-1}\left[\left(\int_\Omega x^\alpha dx\right)^{\frac{p}{q}} \left|\int_\Omega u(x)x^\alpha dx\right|^p + K_2\|u^{(m)}\|^p_{p,\Omega,x^\beta}\right] \\
&\le K_1\left(\left|\int_\Omega u(x)x^\alpha dx\right|^p + \|u^{(m)}(x)\|^p_{p,\Omega,x^\beta}\right).
\end{aligned}$$

where

$$K_1 = 2^{p-1}\max\left\{K_2, \left(\int_\Omega x^\alpha dx\right)^{-\frac{p}{q}}\right\}.$$

LEMMA2.3. (The Hardy inequality, see [7] (3.33)) Let $1 \le p < \infty, a > 0, \Omega = (a,\infty), \alpha, \beta \in R, \beta > p-1, \alpha \le \beta - p, x^\alpha \in L^1(\Omega)$. Let $u(x) \in AC(\Omega)$ and $\lim_{x\to\infty} u(x) = 0$. Then there is constant K such that

$$\int_\Omega |u(x)|^p x^\alpha dx \le K\int_\Omega |u'(x)|^p x^\beta dx.$$

3 THE WEIGHTED POINCARÉ INEQUALITY

THEOREM3.1. Let $1 \le p < \infty, 0 \le m < \infty, a > 0, \Omega = (a,\infty), \alpha, \beta \in R$ and $x^\alpha \in L^1(\Omega)$. Assume that:

(1) $\beta > p-1, \alpha \le \beta - p$;

(2) $u(x) \in W^{m,p}(\Omega; x^\alpha, x^\beta)$ such that

$$u^{(j)}(x) \in AC(\Omega), and, u^{(j)}(x) \to 0 (x \to \infty), 0 < j < m.$$

Then there exists a constant K such that

$$\int_\Omega |u(x)|^p x^\alpha dx \le K\left(\left|\int_\Omega u(x)x^\alpha dx\right|^p + \|u^{(m)}(x)\|^p_{p,\Omega,x^{\beta+(m-1)p}}\right).$$

PROOF. We have from [7, example 5.4] that there exists a constant K_1 such that

$$\inf_{c\in R}\|u(x) - c\|_{p,\Omega,x^\alpha} \le K_1\|u'(x)\|_{p,\Omega,x^\beta}. \tag{3.1}$$

Since

$$u^{(j)}(x) \in AC(\Omega), \quad and, \quad u^{(j)}(x) \to 0(x\to\infty), \quad 0 < j < m,$$

by Lemma 2.3, then there exist the constants $K_2, K_3, \cdots, K_m$ such that

$$\|u'(x)\|_{p,\Omega,x^\beta} \le K_2\|u^{(2)}(x)\|_{p,\Omega,x^{\beta+p}} \le \cdots \le K_m\|u^{(m)}(x)\|_{p,\Omega,x^{\beta+(m-1)p}}.$$

By (3.1), we obtain

$$\inf_{c\in R}\|u(x) - c\|_{p,\Omega,x^\alpha} \le K_1K_m\|u^{(m)}(x)\|_{p,\Omega,x^{\beta+(m-1)p}}.$$

According to Lemma 2.2, there exists a constant K such that

$$\int_\Omega |u(x)|^p x^\alpha dx \le K\left(\left|\int_\Omega u(x)x^\alpha dx\right|^p + \|u^{(m)}(x)\|^p_{p,\Omega,x^{\beta+(m-1)p}}\right).$$

THEOREM3.2. Let $1 \le p < \infty, 0 \le m < \infty$ and $\Omega = [0, \infty)$. Assume that:
(1) $w(x), u^{(j)}(x)(0 \le j < m)$ are locally absolutely continuous on Ω;
(2) $u(0) = 0, w(x) > 0$, and $\lambda w(x) \le (-1)^m w^{(m)}(x)$ on Ω for some $\lambda > 0$;
(3) $|u(x)|^p w^{(m-1)}(x) \to 0 (x \to \infty)$;
$|u(x)|^{p-1}|u^{(k)}(x)|w^{(m-k-1)}(x) \to 0(x \to \infty), 1 \le k < m$;
(4) there exist positive constants $C_1, C_2, \cdots, C_{m-1}$ such that

$$C_k \mid u(x) \|\, u^k(x) \mid w(x) \ge \mid u'(x) \|\, u^k(x) \mid w^{m-k-1}, 1 \le k < m.$$

Then

$$\int_\Omega |u(x)|^p w(x)dx \le \left(\frac{p + p(p-1)\Sigma_{k=1}^{m-1} C_k}{\lambda}\right)^p \int_\Omega |u^{(m)}(x)|^p w(x)dx. \tag{3.2}$$

PROOF. We suppose without loss of generality that $\int_\Omega |u(x)|^p w(x)dx > 0$, $\int_\Omega |u^{(m)}(x)|^p w(x)dx < \infty$ and m is an odd number.Let N be such that$\int_0^n |u(x)|^p w(x)dx > 0$ for all $n \ge N$. Then in view of (1) and (2) we may for each $n \ge N$ apply integration by parts on $[0, n]$ to obtain

$$\begin{aligned}
&\lambda \int_0^n |u(x)|^p w(x)dx \\
&\le -\int_0^n |u(x)|^p w^{(m)}(x)dx \\
&= -|u(n)|^p w^{(m-1)}(n) + p\int_0^n |u(x)|^{p-1}|u'(x)|w^{(m-1)}(x)dx \\
&= -|u(n)|^p w^{(m-1)}(n) + p|u(n)|^{p-1}|u'(n)|w^{(m-2)}(n) \\
&\quad - p(p-1)\int_0^n |u(x)|^{p-2}|u'(x)|^2 w^{(m-2)}(x)dx \\
&\quad - p\int_0^n |u(x)|^{p-1}|u^{(2)}(x)|w^{(m-2)}(x)dx \\
&= \cdots
\end{aligned}$$

$$\begin{aligned}
&= -|u(n)|^p w^{(m-1)}(n) + p|u(n)|^{p-1}|u'(x)|w^{(m-2)}(n) \\
&\quad - p|u(n)|^{p-1}|u^{(2)}(n)|w^{(m-3)}(n) + \cdots - p|u(n)|^{p-1}|u^{(m-1)}(n)|w(n) \\
&\quad - p(p-1)\int_0^n |u(x)|^{p-2}|u'(x)|^2 w^{(m-2)}(x)dx \\
&\quad + p(p-1)\int_0^n |u(x)|^{p-2}|u'(x)||u^{(2)}(x)|w^{(m-3)}(x)dx \\
&\quad - \cdots + p(p-1)\int_0^n |u(x)|^{p-2}|u'(x)||u^{(m-1)}(x)|w(x)dx \\
&\quad + p\int_0^n |u(x)|^{p-1}|u^{(m)}(x)|w(x)dx.
\end{aligned} \tag{3.3}$$

If $1 < p < \infty$, then by Hölder's inequality we have

$$\int_0^n |u(x)|^{p-1}|u^{(m)}(x)|w(x)dx \le \left[\int_0^n |u^{(m)}(x)|^p w(x)dx\right]^{\frac{1}{p}} \left[\int_0^n |u(x)|^p w(x)dx\right]^{\frac{1}{q}}. \tag{3.4}$$

By (3.3), (3.4), condition (3),(4) and the Monotone Convergence Theorem, we have

$$\lambda \left[\int_\Omega |u(x)|^p w(x)dx\right]^{\frac{1}{p}} \le (p + p(p-1)\Sigma_{k=1}^{m-1} C_k) \left[\int_\Omega |u^{(m)}(x)|^p w(x)dx\right]^{\frac{1}{p}}.$$

It implies that theorem 3.2 holds for $1 < p < \infty$.

For $p = 1$, then (3.3) shows that

$$\lambda \int_0^n |u(x)|w(x)dx$$
$$\leq -|u(n)|w^{(m-1)}(n) + |u'(n)|w^{(m-2)}(n)$$
$$- |u^{(2)}(n)|w^{(m-3)}(n) + \cdots - |u^{(m-1)}(n)|w(n)$$
$$+ \int_0^n |u^{(m)}(x)|w(x)dx,$$

and we finish the proof of theorem 3.2.

EXAMPLE3.3. Let $1 \leq p < \infty, 0 \leq m < \infty$ and $\Omega = [0, \infty)$. Suppose that $w(x) = e^{-2x}$ and

$$u(x) = \begin{cases} x^n, & \text{if} \quad x \in [0,1], \\ e^{x-1}, & \text{if} \quad x \in (1, \infty). \end{cases}$$

If $n \geq m$,then $u(x)$ and $w(x)$ satisfy all of conditions (1),(2),(3) and(4) in Theorem 3.2.So there exist $C_1, C_2, \cdots, C_{m-1}$ such that

$$\int_\Omega |u(x)|^p w(x)dx \leq \left(\frac{p + p(p-1)\Sigma_{k=1}^{m-1} C_k}{\lambda}\right)^p \int_\Omega |u^{(m)}(x)|^p w(x)dx. \tag{3.5}$$

COROLLARY3.4. Let $1 \leq p < \infty, 0 \leq m < \infty$ and $\Omega = (-\infty, 0]$. Assume that:
(1) $w(x), u^{(j)}(x)(0 \leq j < m)$ are locally absolutely continuous on Ω;
(2) $u(0) = 0, w(x) > 0$, and $\lambda w(x) \leq (-1)^m w^{(m)}(x)$ on Ω for some $\lambda > 0$;
(3) $|u(x)|^p w^{(m)}(x) \to 0(x \to -\infty)$;
$|u(x)|^{p-1}|u^{(k)}(x)|w^{(m-k-1)}(x) \to 0(x \to -\infty), 1 \leq k < m$;
(4) there exist positive constants $C_1, C_2, \cdots, C_{m-1}$ such that

$$C_k \mid u(x) \mid\mid u^k(x) \mid w(x) \geq \mid u^{'}(x) \mid\mid u^k(x) \mid w^{m-k-1}, 1 \leq k < m.$$

Thus

$$\int_\Omega |u(x)|^p w(x)dx \leq \left(\frac{p + p(p-1)\Sigma_{k=1}^{m-1} C_k}{\lambda}\right)^p \int_\Omega |u^{(m)}(x)|^p w(x)dx. \tag{3.6}$$

The proof is analogous to that of Theorem 3.2.A similar result can be presented as following.

COROLLARY3.5. Let $1 \leq p < \infty, 0 \leq m < \infty$ and $\Omega = (-\infty, \infty)$. Assume that:
(1) $w(x), u^{(j)}(x)(0 \leq j < m)$ are locally absolutely continuous on Ω;
(2) $w(x) > 0$, and $\lambda w(x) \leq (-1)^m w^{(m)}(x)$ on Ω for some $\lambda > 0$;
(3) $|u(x)|^p w^{(m)}(x) \to 0(x \to \infty)$;
$|u(x)|^{p-1}|u^{(k)}(x)|w^{(m-k-1)}(x) \to 0(x \to \infty), 1 \leq k < m$;
(4) there exist positive constants $C_1, C_2, \cdots, C_{m-1}$ such that

$$C_k \mid u(x) \mid\mid u^k(x) \mid w(x) \geq \mid u^{'}(x) \mid\mid u^k(x) \mid w^{m-k-1}, 1 \leq k < m.$$

We have

$$\int_\Omega |u(x)|^p w(x)dx \leq \left(\frac{p + p(p-1)\Sigma_{k=1}^{m-1} C_k}{\lambda}\right)^p \int_\Omega |u^{(m)}(x)|^p w(x)dx. \tag{3.7}$$

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On Hopf Cyclicity of Planar Systemswith Multiple Parameters

Abstract. In this paper, we discuss the maximal number of limit cycles which appear under perturbations in Hopf bifurcations by using degenerate first-order Melnikov function with multiple parameters.

Key Words— limit cycle, Hopf bifurcation, cyclicity, Melnikov function.

1 INTRODUCTION

We shall consider a plane system of the form

$$\begin{aligned} \dot{x} &= H_y + \epsilon p(x, y, \epsilon, \delta) \\ \dot{y} &= -H_x + \epsilon q(x, y, \epsilon, \delta) \end{aligned} \tag{1.1}$$

where ϵ is a small parameter, $\delta \in D \subset R^n$ with D bounded,$n \geq 1$,and H,p,q are C^∞ functions.Suppose the unperturbed Hamilton system

$$\begin{aligned} \dot{x} &= H_y \\ \dot{y} &= -H_x \end{aligned} \tag{1.2}$$

has a family of periodic orbits $\{L_h : h \in J\}$ with J an open internal.We may suppose $J = (0, h_0)$ and the limit $\lim\limits_{h \to o} L_h = L_0$ exists and is an elementary center.If the limit $\lim\limits_{h \to h_0} L_h = L_{h_0}$ exists finitely,it is usually a separatrix cycle.For Eq.(1.1) the problem is to find the maximal number of limit cycles in a given neighborhood of the union $\bigcup\limits_{0 \leq h \leq h_0} L_h$. For details of some of this work we refer to Bautin [1],Blows and Lloyd [2],Cima,Gasull and Manoss [3],Han [4,5,6],Han and Chen [7],Han,Ye and Zhu [8],Han and Zhang [9],Han and Zhu [10],Li and Zhang [11],Lins,De Melo and Pugh [12],Lloyd and Lynch [13],Luo,Wang,Zhu and Han [14],Mardesic [15].

In this paper,we consider the problem of cyclicity in the Hopf bifurcation for general system (1.1) with multiple parameters.We suppose that Eq.(1.2) has an elementary center at the origin and that

$$H(x, y) = K(x^2 + y^2) + O(|x, y|^3), \quad x^2 + y^2 \ll 1, K > 0 \tag{1.3}$$

and that the periodic orbits L_h are level curves of H.

In studying,we use a return map of the form

$$P(h, \epsilon, \delta) - h = M(h, \delta)\epsilon + M_2(h, \delta)\epsilon^2 + \cdots,$$

where

$$M(h, \delta) = \oint\limits_{L_h} (H_y q + H_x p) dt|_{\epsilon=0}, \tag{1.4}$$

we call $M(h, \delta)$ the first-order Melnikov function of system(1.1).

2 THE MAIN RESULTS AND PROOF

THEOREM 2.1. Suppose the C^∞ system (1.1) satisfies Eq.(1.3).Let

$$M(h, \delta) = b_0(\delta)h + b_1(\delta)h^2 + \cdots + b_k(\delta)h^{k+1} + O(h^{k+2}), 0 < h \ll 1, \tag{2.1}$$

and

$$\begin{aligned} &b_j(\delta_0) = 0, j = 0, 1, \cdots, k, \\ &det\frac{\partial(b_{i0}, \cdots, b_{ik})}{\partial(\delta_1, \cdots, \delta_{k+1})}(\delta_0) = 0, i = 0, 1, \cdots, s-1, \\ &det\frac{\partial(b_{s0}, \cdots, b_{sk})}{\partial(\delta_1, \cdots, \delta_{k+1})}(\delta_0) \neq 0, \end{aligned} \tag{2.2}$$

where $\delta_0 \in D, \delta_0 = (\delta_{10}, \cdots, \delta_{n0}), n > k \geq 1, b_{ij} = \frac{\partial^i b_j}{\partial \delta_j^i}, i = 0, 1, \cdots, s, j = 0, 1, \cdots, k$,and s is a nonnegative integer.If there exist a vector valued function $\phi(\epsilon, \delta_{k+2}, \cdots, \delta_n) \in R^{k+1}$ such that

Eq.(1.1) has a center at the origin for $(\delta_1,\cdots,\delta_{k+1}) = \phi(\epsilon,\delta_{k+2},\cdots,\delta_n)$, then Eq.(1.1) has at most k limit cycles near the origin for $\delta \in D$ and $|\epsilon| + |\delta - \delta_0|$ sufficiently small,and k limit cycles can appear for some (ϵ,δ). In other words,(1.1) has cyclicity k at the origin for $\delta \in D$ and $|\epsilon| + |\delta - \delta_0|$ sufficiently small.

PROOF.Let u(t,c) denote the solution of Eq.(1.2) with the initial value (c,o).From Eq.(1.3),we have

$$H(c,0) = c^2(K + S(c)), S(0) = 0, S \in C^\infty.$$

Let $r(c) = c\sqrt{K + S(c)}$.Then $r \in C^\infty$ in c.Since H is a first integral of (1.2),we have

$$H(u(t,c)) \equiv r^2(c), t \in R.$$

Let c=c(r) is a unique inverse of r=r(c) and v(t,r)=u(t,c(r)).Then $H(x,y) = r^2$ defines a closed curve L_r which is a periodic orbit of Eq.(1.2) having period T(r).Thus we have

$$H(v(t,r)) \equiv r^2, v(T(r),r) = (c(r),0).$$

Obviously,v(t,r) and T(r) are C^∞ functions and $T(0) > 0$.Let

$$G(\theta,r) = v(\frac{T(r)\theta}{2\pi}, r).$$

Then G is C^∞ and 2π-periodic in θ.By [10,Lemma 5.14] or [14,Lemma 4.4.3],the transformation $(x,y) = G(\theta,r)$ carries Eq.(1.1) into the system

$$\dot{\theta} = \frac{2\pi}{T(r)}\left[1 + \frac{\epsilon G_r \wedge (p(G,\epsilon,\delta), q(G,\epsilon,\delta))}{G_r \wedge (H_y(G), -H_x(G))}\right],$$
$$\dot{r} = \frac{\epsilon}{2r} DH(G)\cdot(p(G,\epsilon,\delta), q(G,\epsilon,\delta))^T. \tag{2.3}$$

By $H(G) = r^2$,we have $DH(G)G_r = 2r$.Using (1.3),we have $G_r \wedge (H_y(G), -H_x(G)) = \pm 2r$.Noting that $p(G,\epsilon,\delta), q(G,\epsilon,\delta) = O(r)$,Eq.(2.3) is C^∞.Then we obtain the following C^∞ 2π-periodic equation

$$\frac{dr}{d\theta} = \epsilon R(\theta,r,\epsilon,\delta) \tag{2.4}$$

where

$$R(\theta,0,\epsilon,\delta) = 0, R(\theta,r,0,\delta) = \frac{T(r)}{4\pi r} DH(G)\cdot(p(G,0,\delta), q(G,0,\delta))^T. \tag{2.5}$$

It is clear that system (1.1) has a limit cycle near the origin if and only if the cylinder equation (2.4) has two 2π-periodic solutions near r=0(one is positive,the other is negative).

Let $P(r,\epsilon,\delta)$ denote the Poincaré map of Eq.(2.4).Then we can write

$$P(r,\epsilon,\delta) = r + \epsilon r F(r,\epsilon,\delta), \tag{2.6}$$

where

$$rF(r,0,\delta) = \int_0^{2\pi} R(\theta,r,0,\delta)d\theta \equiv R_0(r,\delta).$$

Since $\dot{\theta} > 0$ for ϵ small,the origin is stable (resp.,unstable) for Eq.(1.1) if and only if the zero solution of Eq.(2.4) is stable (resp.,unstable).We have

$$r[P(r,\epsilon,\delta) - r] = \epsilon r^2 F(r,\epsilon,\delta) \geq 0 (or \leq 0), for\ all\ r\ small \tag{2.7}$$

for fixed (ϵ,δ)

By [6,Proof of Theorem 1.2],we obtain the relation between the function $R_0(r,\delta)$ and $M(h,\delta)$:

$$M(h,\delta) = rR_0(r,\delta),$$

for $r = \sqrt{h}, h \geq 0$. From Eq.(2.1) we have

$$R_0(r,\delta) = r\left[\sum_{j=0}^{k} b_j(\delta)r^{2j} + O(r^{2k+2})\right].$$

Note that the Poincaré map $P(r,\epsilon,\delta)$ is C^∞ in r.We can write

$$F(r,\epsilon,\delta)=\sum_{j=0}^{2k+1}C_j(\epsilon,\delta)r^j+r^{2k+2}Q(r,\epsilon,\delta), \tag{2.8}$$

where Q is C^∞ in r and

$$C_{2j}(0,\delta)=b_j(\delta),C_{2j+1}(0,\delta)=0,j=0,1,\cdots,k.$$

By Eq.(2.2),the equations $b_{sj}=C_{2j}(\epsilon,\delta),j=0,1,\cdots,k$,have a vector valued solution of the form

$$(\delta_1,\cdots,\delta_{k+1})=\bar{\phi}(\epsilon,b_{s0},\cdots,b_{sk},\delta_{k+2},\cdots,\delta_n) \tag{2.9}$$

for $\delta\in D$ and $|\epsilon|+|\delta-\delta_0|$ sufficiently small.Since Eq.(1.1) has a center at the origin for $(\delta_1,\cdots,\delta_{k+1})=\phi(\epsilon,\delta_{k+2},\cdots,\delta_n)$,we have $P(r,\epsilon,\delta)-r=0$ and especially $C_{2j}(\epsilon,\delta)=0,j=0,1,\cdots,k$.The implicity function theorem implies that

$$\bar{\phi}(\epsilon,b_{s0},\cdots,b_{sk},\delta_{k+2},\cdots,\delta_n)=\phi(\epsilon,\delta_{k+2},\cdots,\delta_n)\Leftrightarrow b_{sj}=0,j=0,1,\cdots,k. \tag{2.10}$$

Substituting Eq.(2.9) into Eq.(2.8) yields that

$$\begin{aligned}F(r,\epsilon,\delta)&=\sum_{j=0}^{k}\left[b_{sj}r^{2j}+C_{2j+1}(\epsilon,\bar{\phi},\delta_{k+2},\cdots,\delta_n)r^{2j+1}\right]+r^{2k+2}Q(r,\epsilon,\bar{\phi},\delta_{k+2},\cdots,\delta_n)\\&\equiv F^*(r,\epsilon,\mu),\end{aligned} \tag{2.11}$$

where $\mu=(b_{s0},\cdots,b_{sk},\delta_{k+2},\cdots,\delta_n)$. Then from Eq.(2.10) and our assumption,we have that $F^*(r,\epsilon,\mu)=0$ for $b_{sj}=0,j=0,1,\cdots,k$.Hence we can write

$$\begin{aligned}C_{2j+1}(\epsilon,\bar{\phi},\delta_{k+2},\cdots,\delta_n)&=\epsilon\sum_{i=0}^{k}b_{si}A_{ij}(\epsilon,\mu),\\Q(r,\epsilon,\bar{\phi},\delta_{k+2},\cdots,\delta_n)&=\sum_{i=0}^{k}b_{si}Q_i(r,\epsilon,\mu).\end{aligned} \tag{2.12}$$

By (2.7) and (2.11),we have

$$C_{2j+1}(\epsilon,\bar{\phi},\delta_{k+2},\cdots,\delta_n)=0\quad as\ b_i=0,i=0,1,\cdots,j,\ j=0,1,\cdots,k.$$

Thus,from Eq.(2.12),we have

$$A_{ij}=0\quad for\ j+1\le i\le k,\ j=0,1,\cdots,k-1. \tag{2.13}$$

Substituting Eqs.(2.12) and (2.13) into Eq.(2.11),we have

$$F^*(r,\epsilon,\mu)=\sum_{j=0}^{k}b_{sj}r^{2j}P_j(r,\epsilon,\mu), \tag{2.14}$$

where

$$P_j(r,\epsilon,\mu)=1+\epsilon\sum_{i=j}^{k}A_{ji}(\epsilon,\mu)r^{2(i-j)+1}+r^{2(k-j)}Q_j(r,\epsilon,\mu),0\le j\le k.$$

From Eq.(2.11),it is easy to see that F^* can have k positive roots in r for some $(\epsilon,b_{s0},\cdots,b_{sk})$.From Eq.(2.14) and by the technique of Bautin[1],we can prove that F^* has at most k roots in $r>0$.The proof is analogous to that of [6,Theorem 1.3].This completes the proof of Theorem 2.1.

COROLLARY 2.2[6] Suppose the C^∞ system (1.1) satisfies Eq.(1.3).Let

$$M(h,\delta)=b_0(\delta)h+b_1(\delta)h^2+\cdots+b_k(\delta)h^{k+1}+O(h^{k+2}),0<h\ll 1,$$

and

$$
\begin{aligned}
&b_j(\delta_0) = 0, j = 0,1,\cdots,k,\\
&det\frac{\partial(b_0,\cdots,b_k)}{\partial(\delta_1,\cdots,\delta_{k+1})}(\delta_0) \neq 0,
\end{aligned}
$$

where $\delta_0 \in D, \delta_0 = (\delta_{10},\cdots,\delta_{n0}), n > k \geq 1$.If there exist a vector valued function $\phi(\epsilon,\delta_{k+2},\cdots,\delta_n) \in R^{k+1}$ such that Eq.(1.1) has a center at the origin for $(\delta_1,\cdots,\delta_{k+1}) = \phi(\epsilon,\delta_{k+2},\cdots,\delta_n)$, then Eq.(1.1) has at most k limit cycles near the origin for $\delta \in D$ and $|\epsilon| + |\delta - \delta_0|$ sufficiently small,and k limit cycles can appear for some (ϵ,δ). In other words,(1.1) has cyclicity k at the origin for $\delta \in D$ and $|\epsilon| + |\delta - \delta_0|$ sufficiently small.

THEOREM 2.3 Suppose the the C^∞ system (1.1) satisfies Eq.(1.3).Let p,q are polynomials of degree s+1 for δ and b_{sj} of Eq.(2.1) satisfies:

(1)rank $\frac{\partial(b_{s0},\cdots,b_{sk})}{\partial(\delta_1,\cdots,\delta_n)}(\delta_0) = k+1, n > k, \delta_0 \in D, b_{ij} = \frac{\partial^i b_j}{\partial\delta_j^i}, i = 0,1,\cdots,s, j = 0,1,\cdots,k$;

(2)as $b_{sj}(\delta) = 0, j = 0,1,\cdots,k$, Eq.(1.1) has a center at the origin.

Then Eq.(1.1) has at most k limit cycles near the origin for $\delta \in D$ and $|\epsilon| + |\delta - \delta_0|$ sufficiently small,and k limit cycles can appear for some (ϵ,δ). In other words,(1.1) has cyclicity k at the origin for $\delta \in D$ and $|\epsilon| + |\delta - \delta_0|$ sufficiently small.

PROOF.By (1),without loss of generality we may suppose

$$
det\frac{\partial(b_{s0},\cdots,b_{sk})}{\partial(\delta_1,\cdots,\delta_{k+1})}(\delta_0) \neq 0.
$$

Since p,q are polynomials of degree s+1 for δ,$b_j (j = 0,1,\cdots,k)$ are polynomials of degree s+1 for δ.Then the equations $b_{sj} = 0, j = 0,1,\cdots,k$,have a vector valued solution of the form

$$
(\delta_1,\cdots,\delta_{k+1}) = \phi(\delta_{k+2},\cdots,\delta_n).
$$

Then $(\delta_{10},\cdots,\delta_{0,k+1}) = \phi(\delta_{0,k+2},\cdots,\delta_{0n})$. By Theorem 2.1,Eq.(1.1) has cyclicity k at the origin for $|\epsilon| + |\delta - \delta_0|$ sufficiently small.The proof is completed.

2 EXAMPLE

EXAMPLE. Consider the system

$$
\begin{aligned}
\dot{x} &= y - \epsilon[\delta_1^2 x + \delta_2^2 x^2 + (\delta_1 + \delta_2)x^3 + \delta_2 x^4],\\
\dot{y} &= -(x + x^2),
\end{aligned}
\tag{3.1}
$$

where ϵ is a small parameter,$\delta = (\delta_1,\delta_2) \in D \subset R^2$ with D bounded.

Let

$$
H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{3}x^3,
$$

then Eq.(1.3) hold.By Eq.(1.4),we have

$$
\begin{aligned}
M(h,\delta) &= \delta_1^2 M_1(h) + \delta_2^2 M_2(h) + (\delta_1 + \delta_2)M_3(h) + \delta_2 M_4(h),\\
M_i(h) &= \oint_{H=h} x^i dy, i = 1,2,3,4.
\end{aligned}
\tag{3.2}
$$

Note that $x^2 = 2h - y^2 - \frac{2}{3}x^3, 2h - y^2 = x^2 + \frac{2}{3}x^3$,along H=h,we have

$$
\begin{aligned}
M_2(h) &= \oint_{H=h}(2h - y^2 - \frac{2}{3}x^3)dy = -\frac{2}{3}M_3(h),\\
M_4(h) &= \oint_{H=h}[(2h - y^2)^2 - \frac{4}{3}x^3(2h - y^2) + \frac{4}{9}x^6]dy\\
&= \oint_{H=h}[-\frac{4}{3}x^3(x^2 + \frac{2}{3}x^3) + \frac{4}{9}x^6]dy\\
&= -\frac{4}{3}M_5(h) - \frac{4}{9}M_6(h).
\end{aligned}
$$

Inserting the above into Eq.(3.2),we have

$$
\begin{aligned}
M(h,\delta) &= \delta_1^2 M_1(h) - \frac{2}{3}\delta_2^2 M_3(h) + (\delta_1+\delta_2)M_3(h) - \frac{4}{3}\delta_2 M_5(h) - \frac{4}{9}\delta_2 M_6(h) \\
&= \delta_1^2 M_1(h) + (\delta_1+\delta_2 - \frac{2}{3}\delta_2^2)M_3(h) - \frac{4}{3}\delta_2 M_5(h) - \frac{4}{9}\delta_2 M_6(h).
\end{aligned}
$$

By Green's formula and then letting

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we can obtain by [6]

$$
\begin{aligned}
M_{2i+1}(h) &= N_{i0}h^{i+1} + O(h^{i+2}),\ i=0,1,2, \\
M_6(h) &= O(h^4),
\end{aligned}
$$

where

$$N_{i0} = -\frac{(2i+1)2^i}{i+1}\int_0^{2\pi}\cos^{2i}\theta d\theta \neq 0,\ i=0,1,2.$$

Thus we have

$$M(h,\delta) = b_0(\delta)h + b_1(\delta)h^2 + b_2(\delta)h^3 + O(h^4)$$

with that

$$
\begin{aligned}
b_0(\delta) &= N_{00}\delta_1^2 \\
b_1(\delta) &= a_1\delta_1^2 + N_{10}\delta_1 + N_{10}\delta_2 - \frac{2}{3}N_{10}\delta_2^2 \\
b_2(\delta) &= a_2\delta_1^2 + a_3\delta_1 + a_3\delta_2 - a_3\frac{2}{3}\delta_2^2 - \frac{4}{3}N_{20}\delta_2
\end{aligned}
$$

where a_1, a_2, a_3 are constants independent of δ.Then we have

$$b_j(0) = 0,\ j=0,1, \quad det\frac{\partial(b_0,b_1)}{\partial(\delta_1,\delta_2)}(0) = 0, \quad det\frac{\partial(b_{10},b_{11})}{\partial(\delta_1,\delta_2)}(0) \neq 0.$$

By [6,Lemma 3.1],Eq.(3.1) has a center at the origin.By Theorem 2.2,Eq.(3.1) has cyclicity 1 at the origin for $|\epsilon| + |\delta|$ sufficiently small.

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