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Remarks on the unsubsampled wavelet transform and the lifting scheme

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Abstract

The lifting scheme, which is known to be a very useful tool for the wavelet transform, is adapted to the calculation of the unsubsampled wavelet coefficients. It is shown that – as in the subsampled case – the two-band transform for each FIR filter pair without common zeros can be performed by a finite number of lifting steps. Inverting each step, one gets a perfect reconstruction equivalent to the arithmetic mean of the result of two different reconstruction filters. Because of this easy inversion, the lifting scheme offers a great flexibility in treating boundary conditions in the case of finitely many data. © 1998 Elsevier Science B.V. All rights reserved.

Zusammenfassung

Das Lifting-Schema, das als ein sehr nützliches Werkzeug für die Wavelet-Transformation bekannt ist, wird an die Berechnung der Wavelet-Koeffizienten ohne "Subsampling" angepaßt. Es wird gezeigt, daß – wie im Fall mit Subsampling – für jedes FIR-Filterpaar ohne gemeinsame Nullstellen die Zwei-Band-Transformation mit einer endlichen Zahl von Lifting-Schritten durchgeführt werden kann. Indem man jeden Schritt invertiert, erhält man eine perfekte Rekonstruktion, die äquivalent ist zum arithmetischen Mittel des Ergebnisses zweier unterschiedlicher Rekonstruktionsfilter. Aufgrund dieser einfachen Invertierung bietet das Lifting-Schema eine große Flexibilität bei der Behandlung von Randbedingungen im Fall endlich vieler Daten. © 1998 Elsevier Science B.V. All rights reserved.

Résumé

Le schéma du soulèvement (lifting scheme), qui est connu comme un outil très utile pour la transformation d'ondelettes, est adapté au calcul des coefficients d'ondelettes sans sous-échantillonnage. Il est démontré que – comme dans le cas du sous-échantillonnage – pour chaque paire de filtres FIR sans zéros communs, la transformation en deux sous-bandes peut être effectuée par un nombre fini de pas de soulèvement (lifting). En invertissant chaque pas de soulèvement, on obtient une reconstruction parfaite qui est équivalente à la moyenne arithmétique de deux filtres de reconstruction différents. Grâce à cette inversion facile, le schéma du soulèvement offre une grande flexibilité pour le traitement de conditions aux limites au cas d'un nombre fini de données. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Wavelet; Filter bank; Lifting scheme; Finite length signals

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1. Introduction

Only wavelets constructed in the framework of a multiresolution analysis (MRA) with scaling function φ will be considered here. We fix the normalization condition $\sum h_k = 1$, such that the *two-scale relation* reads

$$\varphi(x) = 2\sum_{k\in\mathbb{Z}} h_k\varphi(2x-k), \tag{1}$$

and the wavelet fulfills

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} g_k\varphi(2x-k). \tag{2}$$

Let f be a signal. We shall study here the *redundant* coefficients

$$r_i(k) = 2^{-i}\int f(x)\psi(2^{-i}x - 2^{-i}k)\,\mathrm{d}x, \tag{3}$$

$$s_i(k) := 2^{-i}\int f(x)\varphi(2^{-i}x - 2^{-i}k)\,\mathrm{d}x, \tag{4}$$

which fulfill the following recursion formulae:

$$\begin{aligned} s_i(k) &= \sum_{l\in\mathbb{Z}} h_l s_{i-1}(k + 2^{i-1}l), \\ r_i(k) &= \sum_{l\in\mathbb{Z}} g_l s_{i-1}(k + 2^{i-1}l). \end{aligned} \tag{5}$$

Their advantage is a simple behaviour, if the signal is translated (usually referred as translation invariance).

We shall denote the z-transform of a sequence by the corresponding capital letter as for example

$$S_i(z) := \sum_{k\in\mathbb{Z}} s_i(k)z^{-k}. \tag{6}$$

The above recursion formulae then read

$$\begin{aligned} S_i(z) &= H(z^{-2^{i-1}})S_{i-1}(z), \\ R_i(z) &= G(z^{-2^{i-1}})R_{i-1}(z). \end{aligned} \tag{7}$$

If we have recursion filters $\tilde{H}(z)$ and $\tilde{G}(z)$ satisfying the following *condition of perfect reconstruction*:

$$\tilde{H}(z)H(z^{-1}) + \tilde{G}(z)G(z^{-1}) = 1, \tag{8}$$

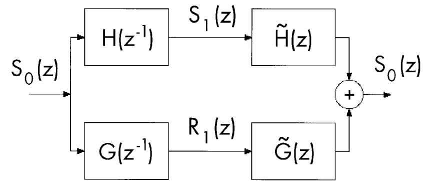

Fig. 1. Perfect reconstruction filter bank (without subsampling).

the coefficients may be successively reconstructed by

$$S_{i-1}(z) = \tilde{H}(z^{2^{i-1}})S_i(z) + \tilde{G}(z^{2^{i-1}})R_i(z). \tag{9}$$

For $i = 1$, this is visualised in Fig. 1.

It is well known that, given any FIR (Finite Impulse Response) filters $H(z)$ and $G(z)$ with no common zeros, Euclid's algorithm always furnishes FIR filters $\tilde{H}(z)$ and $\tilde{G}(z)$ fulfilling Eq. (8).

The reconstruction filters are not unique, we have the following proposition.

Proposition 1. *Suppose that a particular solution* $\tilde{H}_{\text{old}}(z)$ *and* $\tilde{G}_{\text{old}}(z)$ *of Eq. (8) for a fixed pair of analysis filters* $H(z)$ *and* $G(z)$ *is given. Then, using any FIR filter* $T(z)$, *we always get new reconstruction filters by*

$$\tilde{H}_{\text{new}}(z) = \tilde{H}_{\text{old}}(z) + T(z)G(z^{-1}), \tag{10}$$

$$\tilde{G}_{\text{new}}(z) = \tilde{G}_{\text{old}}(z) - T(z)H(z^{-1}), \tag{11}$$

and any FIR reconstruction filters may be constructed in this way, starting from a given pair $\tilde{H}_{\text{old}}(z)$ *and* $\tilde{G}_{\text{old}}(z)$.

That $\tilde{H}_{\text{new}}(z)$ and $\tilde{G}_{\text{new}}(z)$ fulfill Eq. (8) is elementary, conversely the Laurent polynomial $T(z)$ may be constructed analogously to the proof of Proposition 2.

2. Lifting scheme

The lifting scheme is a very powerful tool for the wavelet transform with subsampling, see [1,5–7]. It is shown in this section, how it may be adapted to the unsubsampled case. As the analog of [1], (Theorem 3) we get the following proposition.

Proposition 2 (lifting). *Given any solution $H_{\text{old}}(z)$, $G_{\text{old}}(z)$, $\tilde{H}_{\text{old}}(z)$, $\tilde{G}_{\text{old}}(z)$ of Eq. (8), then for any new solution fulfilling $\tilde{H}_{\text{new}}(z) = \tilde{H}_{\text{old}}(z)$ and $G_{\text{new}}(z) = G_{\text{old}}(z)$, there is an FIR filter $T(z)$ with*

$$H_{\text{new}}(z) = H_{\text{old}}(z) - T(z^{-1})G_{\text{old}}(z), \tag{12}$$

$$\tilde{G}_{\text{new}}(z) = \tilde{G}_{\text{old}}(z) + T(z)\tilde{H}_{\text{old}}(z). \tag{13}$$

Conversely, any FIR filter $T(z)$ defines a new solution by the above equations.

Proof. Let

$$\begin{aligned} U(z) &:= H_{\text{new}}(z) - H_{\text{old}}(z), \\ R(z) &:= \tilde{G}_{\text{new}}(z) - \tilde{G}_{\text{old}}(z). \end{aligned} \tag{14}$$

Subtracting the condition Eq. (8) for the new filters and for the old ones furnishes

$$\tilde{H}_{\text{old}}(z)U(z^{-1}) + R(z)G_{\text{old}}(z^{-1}) = 0. \tag{15}$$

Because of Eq. (8), the greatest common divisor of $\tilde{H}_{\text{old}}(z)$ and $G_{\text{old}}(z^{-1})$ is a Laurent polynomial of degree zero (i.e. of the form $c \cdot z^k$, $k \in \mathbb{Z}$). Therefore $\tilde{H}_{\text{old}}(z)$ divides $R(z)$, and we may construct a Laurent polynomial with the desired properties by the division algorithm

$$T(z) := \frac{R(z)}{\tilde{H}_{\text{old}}(z)}. \tag{16}$$

The converse statement may be verified by elementary calculations. $\square$

The following proposition, which corresponds to [1] (Theorem 4), may be proved analogously.

Proposition 3 (dual lifting). *Given any "old" solution of Eq. (8), for any "new" solution fulfilling $H_{\text{new}}(z) = H_{\text{old}}(z)$ and $\tilde{G}_{\text{new}}(z) = \tilde{G}_{\text{old}}(z)$, there is an FIR filter $T(z)$ with*

$$\tilde{H}_{\text{new}}(z) = \tilde{H}_{\text{old}}(z) + T(z)\tilde{G}_{\text{old}}(z), \tag{17}$$

$$G_{\text{new}}(z) = G_{\text{old}}(z) - T(z^{-1})H_{\text{old}}(z). \tag{18}$$

Conversely, any FIR filter $T(z)$ defines a new solution by the above equations.

We shall now describe the modifications of the factoring algorithm in [1], which allow to express the transform by an arbitrary FIR filter pair $H(z)$, $G(z)$ without common zeros (except possibly at zero or infinity) as a product of lifting and dual lifting steps. Thereby one starts from a particular simple filter pair which – in the unsubsampled case – is connected with the "lazy" wavelet. Here, the "lazy" transform we propose corresponds to the trivial filters

$$H_{\text{L}}(z) = G_{\text{L}}(z) = \tilde{H}_{\text{L}}(z) = \tilde{G}_{\text{L}}(z) = 1/\sqrt{2}. \tag{19}$$

Those factors can be put together in a final factor $\frac{1}{2}$ after the addition step as is shown in the scheme of Fig. 2.

In our description we shall suppose that the degree of the Laurent polynomial $H(z)$ is greater than or equal to the degree of $G(z)$ (if this is not the case, one has simply to reverse the role of the two filters). Euclid's algorithm has to be performed, starting with $A_0(z) := H(z^{-1})$ and $B_0(z) := G(z^{-1})$. Then $Q_{i+1}(z)$ is defined by the division algorithm as the quotient polynomial of $A_i(z)$ by $B_i(z)$, and $B_{i+1}(z)$ as the remainder, such that

$$A_i(z) = B_i(z)Q_{i+1}(z) + B_{i+1}(z), \tag{20}$$

and such that the degree of the remainder $B_{i+1}(z)$ is smaller than that of $B_i(z)$. Furthermore, we define $A_{i+1}(z) := B_i(z)$, and this is continued until $B_i(z) = 0$. Let $n = i$ be the first index i when this happens (except for taking capitals, we kept the notation of [1]). With

$$V(z) := \prod_{i=n}^{1} \begin{bmatrix} 0 & 1 \\ 1 & -Q_i(z) \end{bmatrix}, \tag{21}$$

we have

$$\begin{bmatrix} A_n(z) \\ 0 \end{bmatrix} = V(z)\begin{bmatrix} A_0(z) \\ B_0(z) \end{bmatrix} = V(z)\begin{bmatrix} H(z^{-1}) \\ G(z^{-1}) \end{bmatrix}. \tag{22}$$

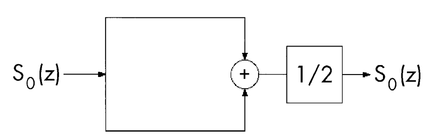

Fig. 2. "Lazy" transform in the unsubsampled case.

By hypothesis, the greatest common divisor $A_n(z)$ has degree zero, and the division may be performed such that this is a constant, i.e. $A_n(z) = c$. We denote the matrix elements of $\boldsymbol{V}(z)$ by $V_{ij}(z)$. Because of Eq. (22), we get a reconstruction filter pair by

$$\tilde{H}_0(z) := \frac{1}{c} V_{11}(z), \qquad \tilde{G}_0(z) := \frac{1}{c} V_{12}(z). \tag{23}$$

Using Eq. (22) furnishes for the z-transform of the coefficients

$$\begin{bmatrix} S_1(z) \\ R_1(z) \end{bmatrix} = \boldsymbol{V}^{-1}(z) \begin{bmatrix} c \cdot S_0(z) \\ 0 \end{bmatrix}, \tag{24}$$

where $\boldsymbol{V}^{-1}(z)$ denotes the inverse matrix of $\boldsymbol{V}(z)$. The matrix $\boldsymbol{V}^{-1}(z)$ has to be factored as in [1], but here the last factor has to be changed. In the case of even n, we introduce $\boldsymbol{W}_e(z)$ as an abbreviation for the first $n-2$ factors of $\boldsymbol{V}^{-1}(z)$:

$$\boldsymbol{W}_e(z) := \prod_{i=1}^{\frac{1}{2}n-1} \begin{bmatrix} 1 & Q_{2i-1}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Q_{2i}(z) & 1 \end{bmatrix} \tag{25}$$

(and $\boldsymbol{W}_e(z) := \boldsymbol{I}$, the 2×2 identity matrix, if $n = 2$). We get the following factorization in lifting and dual lifting steps:

$$\begin{bmatrix} S_1(z) \\ R_1(z) \end{bmatrix} = \boldsymbol{W}_e(z) \begin{bmatrix} 1 & Q_{n-1}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Q_n(z) - 1 & 1 \end{bmatrix} \begin{bmatrix} cS_0(z) \\ cS_0(z) \end{bmatrix}. \tag{26}$$

In the case of odd n, one has to proceed analogously.

The above considerations prove as the analog of [1, Theorem 7] the following proposition.

Proposition 4. *Given any FIR filter pair $H(z)$ and $G(z)$ without common zeros (except possibly at zero or infinity), the calculation of the coefficients $S_1(z)$ and $R_1(z)$ by Eq. (7) can always be done composing a finite number of lifting and dual lifting steps of the form (26).*

This means that the left-hand side of Fig. 1 may – in the case of an even number of factors obtained by Euclid's algorithm – be replaced by the scheme shown in Fig. 3 (the modification for the odd case is obvious). As

$$\begin{bmatrix} 1 & Q(z) \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -Q(z) \\ 0 & 1 \end{bmatrix}, \tag{27}$$

the reconstruction using the lifting scheme becomes trivial. We simply have to reverse the sign of all the Laurent polynomials, and to multiply the final result by c^{-1} (to balance the constant) and $\frac{1}{2}$ (from the lazy transform), in order to get the reconstructed signal. This is shown in Fig. 4.

Therefore, the factorization of the analysis filters into lifting steps immediately furnishes reconstruction filters. One may put the lifting and dual lifting steps of Fig. 4 together, and express them by reconstruction filters $\tilde{H}(z)$ and $\tilde{G}(z)$, as it corresponds to the right-hand side of Fig. 1. Let us have a closer look, which reconstruction filters one gets by this procedure.

If we take the inverses of the matrices of the right-hand side of Eq. (26), only the first factor differs from the product, which defines the matrix $\boldsymbol{V}(z)$ in Eq. (21). A little calculation furnishes for even n

$$\begin{bmatrix} S_0(z) \\ S_0(z) \end{bmatrix} = \frac{1}{c} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{V}(z) \begin{bmatrix} S_1(z) \\ R_1(z) \end{bmatrix}, \tag{28}$$

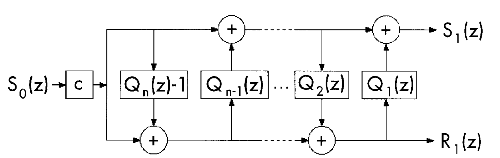

Fig. 3. Calculating the unsubsampled wavelet coefficients using the lifting scheme (in the case of an even number of factors obtained by Euclid's algorithm).

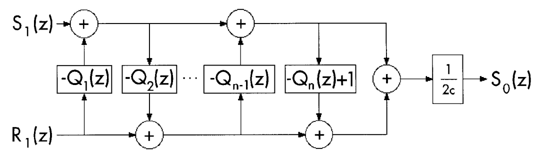

Fig. 4. Reconstruction with the lifting scheme (in the case of an even number of factors obtained by Euclid's algorithm).

and for odd n

$$\begin{bmatrix} S_0(z) \\ S_0(z) \end{bmatrix} = \frac{1}{c}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} V(z) \begin{bmatrix} S_1(z) \\ R_1(z) \end{bmatrix}. \tag{29}$$

In both cases we get a vector whose two components agree. Comparison of the two different expressions for the components on the right hand side for even and odd n shows, that they are just interchanged. It turns out that one component is calculated by the reconstruction filters furnished by Euclid's algorithm (23). The other is calculated by the following new reconstruction filters:

$$\tilde{H}_1(z) = \tilde{H}_0(z) + \frac{1}{c} V_{21}(z), \tag{30}$$

$$\tilde{G}_1(z) = \tilde{G}_0(z) + \frac{1}{c} V_{22}(z). \tag{31}$$

Our choice of the lazy transform in Fig. 2 means that we take the arithmetic mean of the data reconstructed by the two different filters (with index 0 and 1). This might have some stabilizing effect, if a manipulation of the coefficients takes place (for instance in the case of denoising). The resulting equivalent filter bank is shown in Fig. 5. However, to save calculation time, one may simply take $1/c$ times one of the terms before the last summation in Fig. 4, and gets $s_0(k)$ as well.

The lifting scheme has a remarkable property for filters satisfying

$$H(z) + G(z) = 1. \tag{32}$$

Examples of such filters are explicitly given by [4], (Eq. (5.75)), their properties and advantages in feature extraction are described in [2,3]. For those

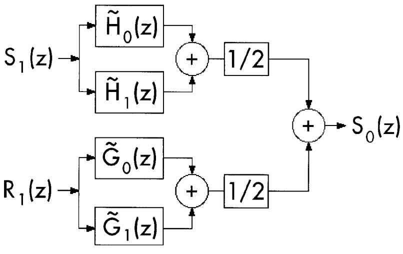

Fig. 5. Filter bank equivalent to the reconstruction by the lifting scheme in Fig. 4.

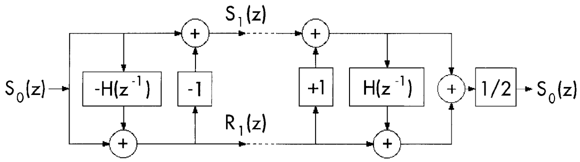

Fig. 6. Analysis and reconstruction with the lifting scheme in the case of filters satisfying Eq. (32).

filters the factorization into lifting steps is trivial. We always get $n = 2$, $c = 1$, $Q_1(z) = -1$ and $Q_2(z) = 1 - H(z^{-1})$. The corresponding lifting steps for analysis and reconstruction are shown in Fig. 6. This scheme furnishes the reconstruction filters $\tilde{H}_0(z) = \tilde{G}_0(z) = 1$, $\tilde{H}_1(z) = H(z^{-1})$ and $\tilde{G}_1(z) = H(z^{-1}) + 1$.

A similar investigation as in [1] shows, that in our case there is no reduction of calculation cost using lifting. The main advantage of lifting in the unsubsampled case will be its great flexibility to adopt boundary conditions in the case of finitely many data, which is discussed in the next section.

Until now in this section, all considerations were restricted to the calculation of $S_1(k)$ and $R_1(k)$, i.e. to the first step in scale. Because of Eq. (7), lifting can also be applied to the general case. For the steps $i > 1$, one simply has to replace all the factors $Q_k(z)$ by $Q_k(z^{2^{i-1}})$.

3. How to treat finite length signals

As the wavelet transform is linear, we shall describe it in this subsection by matrices. Instead of the sequences $s_i(k)$ and $r_i(k)$, we consider vectors $\boldsymbol{s}_i$ and $\boldsymbol{r}_i \in \mathbb{R}^N$ and identify $\boldsymbol{s}_0 = (x_0, x_1, \ldots, x_{N-1})$ with the N original data points. We may put the vectors $\boldsymbol{s}_1$ and $\boldsymbol{r}_1$ together to a single vector in $\mathbb{R}^{2N}$. Convolution operators described by matrices, as they appear on the right hand side of (25), which correspond to lifting or dual lifting steps, will here become $2N \times 2N$ matrices. They have a block structure of the following form ($\boldsymbol{I}$ denotes the $N \times N$ identity matrix):

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{Q} & \boldsymbol{I} \end{bmatrix} \quad \text{respectively} \quad \begin{bmatrix} \boldsymbol{I} & \boldsymbol{Q} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}.$$

In the infinite case, the operator $\boldsymbol{Q}$ corresponds to a convolution with a sequence $q(k)$ whose z-transform is $Q(z)$, i.e. it can be considered as an infinite matrix with matrix elements

$$Q_{ik} = q(i-k). \tag{33}$$

As

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{Q} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{Q} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \tag{34}$$

we have total freedom, in choosing a finite matrix – here again denoted by $\boldsymbol{Q}$ – to replace the original convolution operator. And at the same time, perfect reconstruction is maintained. Thus in the finite case, $\boldsymbol{Q}$ will be an $N \times N$ matrix with the following structure:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{LL} & \boldsymbol{Q}_{LI} & \boldsymbol{0} \\ \boldsymbol{Q}_{IL} & \boldsymbol{Q}_{II} & \boldsymbol{Q}_{IR} \\ \boldsymbol{0} & \boldsymbol{Q}_{RI} & \boldsymbol{Q}_{RR} \end{bmatrix}. \tag{35}$$

The matrix elements of all the inner submatrices satisfy Eq. (33), as they should agree with the infinite case. We have only to change the boundary matrices $\boldsymbol{Q}_{LL}$ and $\boldsymbol{Q}_{RR}$. Indeed, we may choose them arbitrarily. One of the simplest choices would be, to let all the matrix elements satisfy Eq. (33), which corresponds to zero padding with subsequent projection to $\mathbb{R}^N$ (note that this does not correspond to zero padding for the whole wavelet transform, as can be easily seen by calculating examples).

For an even number of steps in Euclid's algorithm, the wavelet transform using lifting now has the form

$$\begin{bmatrix} \boldsymbol{s}_1 \\ \boldsymbol{r}_1 \end{bmatrix} = c \begin{bmatrix} \boldsymbol{I} & \boldsymbol{Q}_1 \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{Q}_2 & \boldsymbol{I} \end{bmatrix} \cdots \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{Q}_n - \boldsymbol{I} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{s}_0 \\ \boldsymbol{s}_0 \end{bmatrix} \tag{36}$$

(with an obvious modification in the case of odd n). Practically, this means that using lifting, we may choose any boundary condition we want – even different ones for different lifting steps – as far as we take exactly the same boundary condition in the corresponding inverse lifting step for reconstruction.

The above formula concerns the calculation for the scale step $i = 1$. For general $i > 1$, the size of the boundary matrices $\boldsymbol{Q}_{LL}$ and $\boldsymbol{Q}_{RR}$ will be increased by a factor 2^{i-1}. We have to fill in $2^{i-1} - 1$ zeros between the matrix elements, and each row has to be repeated 2^{i-1} times, each time shifted by one position to the right. To obtain perfect reconstruction, one has only to replace the whole block submatrix $\boldsymbol{Q}$ by $-\boldsymbol{Q}$ (respectively $\boldsymbol{Q} - \boldsymbol{I}$ by $-\boldsymbol{Q} + \boldsymbol{I}$).

The lifting scheme makes it therefore easy to implement more sophisticated boundary conditions, as for example polynomial extrapolation. But extension takes place at the boundary at each lifting step, and the overall result does not agree in general with the same kind of extension applied once for the whole wavelet transform.

4. Conclusion

The lifting scheme may be successfully applied to the unsubsampled wavelet transform. The main advantage in this case is an easy inversion which allows a large flexibility to adopt various boundary conditions and to calculate easily the corresponding boundary filters in the case of finite length signals. This leads to algorithms which are easy to implement.

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