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A DELAY TIME MULTI-LEVEL ON-CONDITION PREVENTIVE MAINTENANCE INSPECTION MODEL BASED ON CONSTANT BASE INTERVAL RISK—WHEN INSPECTION DETECTS PENDING FAILURE

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Abstract—The model described in this paper is one of a series, which determines the optimal multi-level inspection–maintenance policy for a stochastically deteriorating multi-state sub-system, using the delay-time concept. The sub-system deterioration is assumed to be a non-decreasing semi-Markov process, where states are self-announced and inspection detects the sign of pending failure. Emphasis is placed on constant availability and reduction of production losses, deterioration rate and subsequent sub-system failure. In this respect, inspection is scheduled in such a way that the risk of failure is a constant for each inspection interval. Two pairs of mathematical models and softwares have been developed, and the policy decisions taken have been based on two criteria for optimisation. These decisions have then been validated by carrying out a simulation exercise using the ProModel simulation package. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Production systems can be viewed as multi-state stochastically deteriorating complex systems. Their parts, from a maintenance point of view, can be grouped into five characteristically different sub-systems. Thus, separate optimum multi-level pseudo-control limit maintenance policies have been proposed [1–5]. This paper deals with a gradually deteriorating sub-system, whose present state is self-announced and inspection detects the sign of pending failure, the transition state, its status and the time of transition. A delay-time concept, first introduced by Christer [6], regards the failure mechanism as a two-stage process. A fault initiates in a sub-system and becomes prominent at time y. This can be identified if inspection is carried out at the time. If the fault is not attended to, the faulty sub-system subsequently changes its state after some further interval h, which Christer called the failure delay time.

Research has been carried out [7–10, 6, 11–18] using this concept in maintenance modelling for two-state single- or multi-component [7, 8, 18] systems, where inspection intervals for perfect and/or imperfect inspections are taken as constant or variable [11], and repair restores the system, taking into account subjective and objective [7, 8, 18] data. It would seem practically unreasonable to consider repair as a renewal of a system to its original condition [19] and constant inspection intervals may not have a constant risk of failure, which results in inconsistent availability, and hence a variable production rate and high inventory, labour and production costs [5]. Thus, the concept of delay time is here extended to multi-level maintenance of a multi-state sub-system, where inspections are scheduled in such a way that the risk of failure is constant for each inspection interval [20, 21].

2. MATHEMATICAL MODEL

Let the deterioration process of a sub-system be a semi-Markov process with state space $\Im=\{1, 2,\ldots,L\}$, where states are described by the level of deterioration and the nature of the process limits the occurrence of transitions to higher states. The present state of the sub-system is self-announcing, whereas inspection detects the sign of pending failure. The

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maintenance policy selected is a pseudo-control limit policy $\delta(\bullet)$, where maintenance action is determined by the control state β. Let $\aleph=\{1, 2,\ldots, \beta-1\}$ be a set of states which asks for no repair action if they are functional, and minimal repair when they are non-functional. Its complementary set $\aleph^c=\{\beta, \beta+1,\ldots, L\}$ is the set of states which calls for repair or replacement of a sub-system to some better state in set $\wp\subseteq\aleph$. Whenever inspection detects pending failure before the next inspection, on-condition preventive maintenance (OCPM) is performed instantaneously. This OCPM does not change the present and transition states and their transition time, but reduces the probability of transition to a non-functional state. Mathematically, such a pseudo-control limit policy $\delta(k_*)$ (where k_* represents either functional state k_f or non-functional state k_{nf}) can be expressed as:

$$\delta(k_*) = \begin{cases} i = k \text{ for } k_*<\beta\in\aleph \text{ and } i\in\aleph \\ i<k_* \text{ for } k_*\geq\beta\in\aleph^c \text{ and } i\in\wp\subseteq\aleph \end{cases} \tag{1}$$

Let the sub-system be entered in state $i\in\wp\subseteq\aleph$, at reference or lead time $T_L(i)$, and inherit manufacturing faults with a probability of $P_i(i)$, or faults may arise later on during use with a probability of $[1-P_i(i)]$, and subsequently cause its transition to any higher state $j\in\Im$, with a probability of $P(i,j)$, where the transition state j may be functional with a probability of $P_f(j)$, and non-functional with a probability of $\{1-P_f(j)\}$. A joint probability density function (pdf) as a result of these faults, which hereafter will be called inherited faults (IF) and deterioration faults (DF), for the transition process is $f(\bullet)$. Let n number of inspections be carried out at an inspection time $T(i,1)$, $T(i,2),\ldots,$ and $T(i,n)$, independent of the nature of the faults and transition states, given that process has started in state i (Fig. 1), where these inspections are scheduled in such a way that the expected risk of transition $A(\bullet)$ between any two consecutive inspections is constant.

The probability of transition of the sub-system $\mathscr{A}_1(i,j,m)$ and its expected time $\mathcal{T}_1(i,j,m)$, during the mth inspection interval $[T(i,m), T(i,m+1))$ (Fig. 2), given that the following set of conditions exists, which hereafter will be called set 1 conditions: (i) the sub-system has entered in state i, at time $T_L(i)$, which contains inherited defects with a probability of

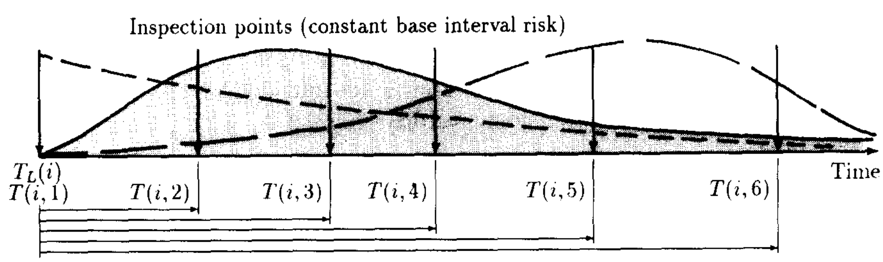

Fig. 1. Joint probability density functions of the transition process for an initial state $j=1$ and transition states $j=2$ (- - -), $j=3$ (— — —) and $j=4$ (——).

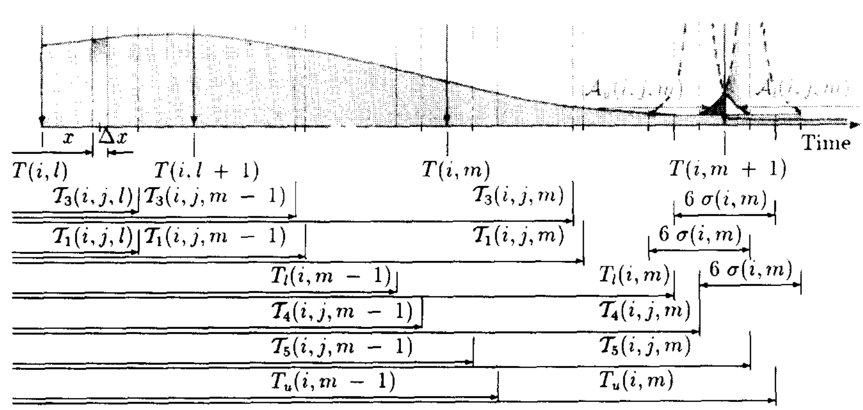

Fig. 2.The probability density functions for delay time (——) and inspection error (— — —) and expected transition times for an inherited fault.

$P_i(i)$, (ii) it has delay time in a small interval $[x, +\Delta x)$, the probability of this event being $\psi(i,j,x)\mathrm{d}x$, and (iii) the sub-system will make the next transition to state j, respectively, are [5]:

$$\mathcal{A}_1(i,j,m) = P_i(i)\int_{T(i,m)}^{T(i,m+1)} \psi(i,j,x)\mathrm{d}x \tag{2}$$

and

$$\mathcal{T}_1(i,j,m) = \left\{\frac{P_i(i)\displaystyle\int_{T(i,m)}^{T(i,m+1)} h\psi(i,j,x)\mathrm{d}x}{\mathcal{A}_1(i,j,m)}\right\} \tag{3}$$

for $1\leq m\leq n-1$, $i\in\aleph$ and $i<j\in\aleph$.

The probability of transition of the subsystem, $A_1(i,j,m,l)$ and its expected time $T_1(i,j,m,l)$ (Fig. 3), during the mth inspection interval $[T(i,m), T(i,m+1))$, provided the following set of conditions exists, which hereafter will be called set 2 conditions: (i) the sub-system has entered in state i, at time $T_L(i)$, (ii) the fault will be initiated within the lth inspection interval $[T(i,l), T(i,l+1))$ in a small interval $[y, y+\Delta y)$, with a probability of $g(i,j,y)\Delta y$, has delay time in a small interval $[y+h, y+h+\Delta h]$, the probability of this event being $\phi(i,j,h)\mathrm{d}h$, and (iii) the sub-system will make the next transition to state j, respectively, are [5]:

$$A_1(i,j,m,l) = \int_{T(i,l)}^{T(i,l+1)}\left\{\int_{[T(i,m)-y]}^{[T(i,m+1)-y]} \phi(i,j,h)\mathrm{d}h\right\}g(i,y)\mathrm{d}y \tag{4}$$

and

$$T_1(i,j,m,l) = \int_{T(i,l)}^{T(i,l+1)}\left[y + \left\{\frac{\displaystyle\int_{[T(i,m)-y]}^{[T(i,m+1)-y]} h\phi(i,j,h)\mathrm{d}h}{A_1(i,j,m,l)}\right\}\right]g(i,y)\mathrm{d}y \tag{5}$$

for $l\leq m$, $1\leq m\leq n-1$, $T(i,l)\leq y<T(i,l+1)$, $i\in\aleph$ and $i<j\in\Im$ where

$$\sum_{m=1}^{n-1}[\mathcal{A}_1(i,j,m) + \sum_{l=1}^{m} A_1(i,j,m,l)] = 1. \tag{6}$$

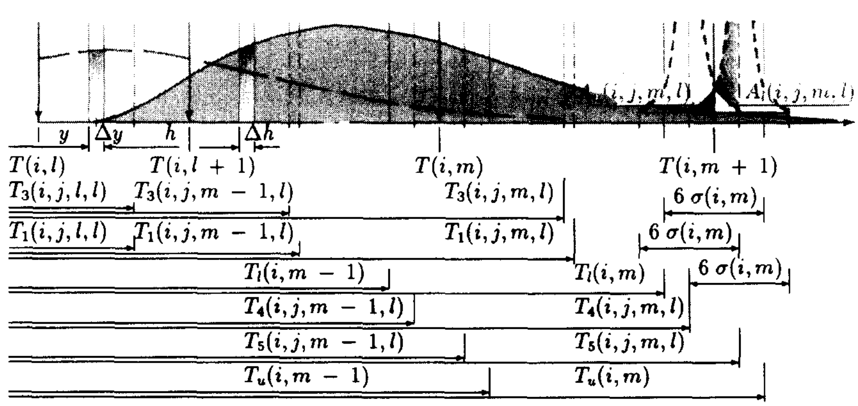

Fig. 3. The probability density functions for fault initiating times, (– – –), delay time (——) and inspection error (- - -) and expected transition times for a deterioration fault.

Thus, for an initial state $i \in \Im$, the expected total constant base interval risk of transition $A(m)$, in any mth interval irrespective of destination state and the nature of the fault, is:

$$A(m) = \sum_{j=i+1}^{L} P(i,j)\left[\mathcal{A}_1(i,j,m) + \sum_{l=1}^{m} A_1(i,j,m,l)\right]. \tag{7}$$

Let the condition or state of the sub-system be taken as a function of multiple parameters. Some of these parameters can be measured directly, whereas others cannot. Thus a fault already present or initiated is reflected in the parameter, which is the measure of its existence with the probability of η. Imperfect inspection may, therefore, detect only the reflected fault with a probability of ξ. Once a fault is detected, inspection detects the status of the transition state with a probability ς, and its transition time with an accuracy of $\pm\zeta$ in fraction of the width of the forthcoming inspection interval. Thus the upper and lower error limits, $T_u(i,m)$ and $T_l(i,m)$ for the estimation of a transition time $T(i,m)<T\leq T(i,m+1)$, at the time of the mth inspection (Fig. 2 and Fig. 3), are:

$$T_u(i,m) = T + \zeta\{T(i,m+1) - T(i,m)\} \tag{8}$$

and

$$T_l(i,m) = T - \zeta\{T(i,m+1) - T(i,m)\} \tag{9}$$

for $1 \leq m < n$ and $i \in \aleph$.

Inspection assessment about the time of transition can be wrong in two ways: (a) assessed as failing before the next inspection (with a probability of $A_u(i,j,m)$ for IF and $A_u(i,j,m)$ for DF) when it would not and (b) assessed as surviving until the next inspection (with a probability of $\mathcal{A}_l(i,j,m,l)$ for IF and $\mathcal{A}_l(i,j,m,l)$ for DF) and then fails, where these probabilities are taken as a function of the initial state i, transition state j, the probability of transition during the forthcoming inspection interval and its width $[T(i,m), T(i,m+1))$, and the fault initiating interval $T(i,l)$, $T(i,l+1)$ [5]. Since the economic impact of the policy decisions is a function of the nature of the fault, the number of inspections and its error, the probability and time of transition, the nature of the policy decision and its consequences, inspection intervals are further divided into sub-intervals (Figs 2 and 3). In this case the probabilities and expected time of transition of the sub-system when it contains inherited faults during intervals $[T(i,m), T_l(i,m))$, $[T_u(i,m-1), T_l(i,m))$, $[T_l(i,m), T(i,m+1))$ and $[T(i,m+1), T_u(i,m))$, given that set 1 conditions exist, are: $\mathcal{A}_2(i,j,m)$, $\mathcal{A}_3(i,j,m)$, $\mathcal{A}_4(i,j,m)$ and $\mathcal{A}_5(i,j,m)$, $\mathcal{T}_2(i,j,m)$, $\mathcal{T}_3(i,j,m)$, $\mathcal{T}_4(i,j,m)$ and $\mathcal{T}_5(i,j,m)$, respectively. The probabilities and expected time of transition of a sub-system, when a fault arises as a result of deterioration of the sub-system, provided that set 2 conditions exist, are: $A_2(i,j,m,l)$, $A_3(i,j,m,l)$, $A_4(i,j,m,l)$, and $A_5(i,j,m,l)$, and $T_1(i,j,m,l)$, $T_2(i,j,m,l)$, $T_3(i,j,m,l)$, $T_4(i,j,m,l)$ and $T_5(i,j,m,l)$, respectively [5].

The probabilities of OCPM $\mathcal{P}_o(i,j,m)$ and $P_o(i,j,m,l)$, at the time of the mth inspection, provided that set 1 and set 2 conditions exist, respectively, are:

$\mathcal{P}_o(i,j,m) =$ (10)

0 for $1 > m$ and/or $m > n-1$, $i \in \aleph$ and $i<j \in \Im$

$\eta\xi\{1-\xi\}^{(m-l)}[\mathcal{A}_2(i,j,m) + \{1-\mathcal{A}_l(i,j,m)\}\mathcal{A}_4(i,j,m)][\varsigma\{1-P_f(j)\} + (1-\varsigma)P_f(j)]$;

for $1 = m = n-1$, $i \in \aleph$ and $i<j \in \Im$

$\eta\xi\{1-\xi\}^{(m-1)}[\mathcal{A}_2(i,j,m) + \{1-\mathcal{A}_l(i,j,m)\}\mathcal{A}_4(i,j,m) + \mathcal{A}_5(i,j,m)\mathcal{A}_u(i,j,m)][\varsigma$

$\{1-P_f(j)\} + (1-\varsigma)P_f(j)]$;for $1 = m<n-1$, $i \in \aleph$ and $i<j \in \Im$ $\sum_{\kappa=1}^{m}$

$\eta\xi\{1-\xi\}^{(\kappa-1)}[\mathcal{A}_5(i,j,m-1)\{1-\mathcal{A}_u(i,j,m-1)\} + \mathcal{A}_3(i,j,m) + \{1-\mathcal{A}_l(i,j,m)\}\mathcal{A}_4(i,j,m)$

$+ \mathcal{A}_5(i,j,m)\mathcal{A}_u(i,j,m)][\varsigma\{1-P_f(j)\} + (1-\varsigma)P_f(j)]$; for $1<m<n-1$, $i \in \aleph$ and $i<j \in \Im$

$\sum_{\kappa=1}^{m} \eta\xi\{1-\xi\}^{(\kappa-1)}[\mathcal{A}_5(i,j,m-1)\{1-\mathcal{A}_u(i,j,m-1)\} + \mathcal{A}_3(i,j,m) + \{1-\mathcal{A}_l(i,j,m)\}\mathcal{A}_4$

$(i,j,m)][\varsigma\{1-P_f(j)\} + (1-\varsigma)P_f(j)]$;for $1<m = n-1$, $i \in \aleph$ and $i<j \in \Im$

and

$$P_o(i,j,m,l) = \tag{11}$$

$$0 \text{ for } l \geq m \text{ and/or } > n-1,\ i \in \aleph \text{ and } i<j \in \Im$$

$$\eta\xi\{1-\xi\}^{(m-l-1)}[A_2(i,j,m,l) + \{1-A_l(i,j,m,l)\}A_4(i,j,m,l)][\varsigma\{1-P_f(j)\} + \{1-\varsigma\}P_f(j)];$$

$$\text{for } l+1 = m = n-1, i \in \aleph \text{ and } i<j \in \Im$$

$$\eta\xi\{1-\xi\}^{(m-l-1)}[A_2(i,j,m,l) + \{1-A_1(i,j,m,l)\}A_4(i,j,m,l) + A_5(i,j,m,l)A_u(i,j,m,l)]$$

$$\begin{cases} [\varsigma\{1-P_f(j)\} + (1-\varsigma)P_f(j)]; \text{for } l+1 = m<n-1,\ i \in \aleph \text{ and } i<j \in \Im \displaystyle\sum_{\kappa=l+1}^{m} \eta\xi \\ \{1-\xi\}^{(\kappa-l-1)}[A_5(i,j,m-1,l)\{1-A_u(i,j,m-1,l)\} \end{cases}$$

$$+ A_3(i,j,m,l) + \{1-A_1(i,j,m,l)\}A_4(i,j,m,l) + A_5(i,j,m,l)A_u(i,j,m,l)][\varsigma\{1-P_f(j)\}$$

$$+ (1-\varsigma)P_f(j)]; \text{for } l+1<m<n-1,\ i \in \aleph \text{ and } i<j \in \Im$$

$$\sum_{\kappa=l+1}^{m} \eta\xi\{1-\xi\}^{(\kappa-l-1)}[A_5(i,j,m-1,l)\{1-A_u(i,j,m-1,l)\} + A_3(i,j,m,l) + \{1-A_1(i,j,m,l)\}$$

$$A_4(i,j,m,l)][\varsigma\{1-P_f(j)\} + (1-\varsigma)P_f(j)]; \text{for } l + 1<m = n-1,\ i \in \aleph \text{ and } i<j \in \Im$$

Let a penalty be charged at a uniform rate of $c_l(i)$ per unit production time lost due to unavailability of the sub-system in state i, independent of the transition state j and the cause of unavailability. The inspection cost $c_i(i)$ per inspection is paid in $\vartheta_i(i)$ instalments, where the νth instalment is equal to $\rho_i(\nu,i)$, a fraction of $c_i(i)$. This is paid at time $\Upsilon_i(\nu,i)$ from the time of the inspection. Thus the net present value (NPV) of the expected single step costs of the inspection plus penalty paid during the time of the inspection T_i, $C_i(i)$, is:

$$C_i(i) = \tag{12}$$

$$\sum_{j=i+1}^{L} P(i,j) \sum_{m=1}^{n-1} \left[\mathcal{A}_1(i,j,m) + \sum_{l=1}^{m} A_1(i,j,m,l) \right]$$

$$\sum_{k=1}^{m} = 1 \left\{ \sum_{\nu=1}^{\vartheta_1(i)} c_i(i)\rho_i(\nu,i)(V_1V_2)^{[T_L(i) + T(i,\kappa) + (\kappa-1)T_1 + \Upsilon_1(\nu,i)]} \right.$$

$$\left. + c_l(i)T_i(V_1V_2)^{[T_L(i) + T(i,\kappa) + (\kappa-0.5)T_i]} \right\}$$

for $i \in \aleph$ and $j \in \Im$, where V_1 V_2 are the annual time value or discount factor and the annual net inflation factor, respectively [7].

Let the OCPM cost $c_o(i)$, be paid in $\vartheta_o(i)$ instalments, where the νth instalment is equal to $\rho_o(\nu,i)$ a fraction of $c_o(i)$, which is paid at time $\Upsilon_o(\nu,i)$ from the time when pending failure is detected. Thus, the NPV of the expected single step costs of OCPM, plus penalty during the time required for OCPM $T_o(i)$, regardless of transition state $C_o(i)$, is:

$$C_o(i) = \tag{13}$$

$$\sum_{j=i+1}^{L} P(i,j) \sum_{m=1}^{n-1} [\mathcal{P}_o(i,j,m) + \sum_{l=1}^{m} P_o(i,j,m,l)]$$

$$\left\{ \sum_{\nu=1}^{\vartheta_o(i)} c_o(i)\rho_o(\nu,i)(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + \Upsilon_o(\nu,i)]} \right.$$

$$\left. + c_l(i)T_o(i)(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + 0.5T_o(i)]} \right\}$$

for $i \in \aleph$ and $j \in \Im$.

Minimal repair is performed at cost rate $c_m(j)$, whenever transition occurs to a non-functional state $j \in \aleph$, which takes time $T_m(j)$. The cost is paid in $\vartheta_m(j)$ instalments, where the νth instalment is equal to $\rho_m(\nu,j)$, a fraction of $c_m(j)$. This is paid in at time $Y_m(\nu,j)$ from the expected time of transition to a non-functional state, plus time taken by inspection. Therefore, the NPV of the single-step expected cost of on-failure minimal repair plus the penalty $C_m(i)$ is:

$$C_m(i) = \sum_{j=i+1}^{\beta-1} P(i,j)\{1-P_f(j)\}\left\{\sum_{m=1}^{n-1}\left[\left[\left\{(1-\eta) + \eta(1-\xi)^m + \sum_{\kappa=1}^{m}\eta\xi(1-\xi)^{(\kappa-1)}\right.\right.\right.\right.$$
$$\left.(1-\varsigma)\right\}\mathscr{A}_1(i,j,m)\left\{\sum_{\nu=1}^{\vartheta_m(j)} c_m(j)\rho_m(\nu,j)(V_1V_2)^{[T_L(i) + mT_i + \mathcal{T}_1(i,j,m) + Y_m(\nu,j)]}\right.$$
$$\left. + c_l(i)T_m(j)(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + \mathcal{T}_1(i,j,m) + 0.5T_m(j)]}\right\} + \sum_{\kappa=1}^{m}\eta\xi(1-\xi)^{(\kappa-1)}\varsigma\mathscr{A}_f(i,j,m)$$
$$\mathscr{A}_4(i,j,m)\left\{\sum_{\nu=1}^{\vartheta_m(j)} c_m(j)\rho_m(\nu,j)\ (V_1V_2)^{[T_L(i) + mT_i + \mathcal{T}_4(i,j,m) + Y_m(\nu,j)]} + c_l(i)T_m(j)\right.$$
$$\left.(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + \mathcal{T}_4(i,j,m) + 0.5T_m(j)]}\right\} + \sum_{l=1}^{m}\left(\left\{(1-\eta) + \eta(1-\xi)^{(m-l)} + \sum_{\kappa=l+1}^{m}\right.\right.$$
$$\left.\eta\xi(1-\xi)^{(\kappa-l-1)}(1-\varsigma)\right\}A_1(i,j,m,l)\left\{\sum_{\nu=1}^{\vartheta_m(j)} c_m(j)\rho_m(\nu,j)(V_1V_2)^{[T_L(i) + mT_i + T_1(i,j,m,l) + Y_m(\nu,j)]}\right.$$
$$\left. + c_l(i)T_m(j)(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + T_1(i,j,m,l) + 0.5T_m(j)]}\right\} + \sum_{\kappa=l+1}^{m}\eta\xi(1-\xi)^{(\kappa-l-1)}\varsigma A_1$$
$$(i,j,m,l)A_4(i,j,m,l)\left\{\sum_{\nu=1}^{\vartheta_m(j)} c_m\ (j)\rho_m(\nu,j)(V_1V_2)^{[T_L(i) + mT_i + T_4(i,j,m,l) + Y_m(\nu,j)]} + c_l(i)T_m(j)\right.$$
$$\left.\left.\left.\left.(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + T_4(i,j,m,l) + 0.5T_m(j)]}\right\}\right)\right]\right\} \tag{14}$$

for $l<m\leq n\text{-}1$, $i \in \aleph$ and $j \in \Im$.

$$C_m(i) = \sum_{j=i+1}^{\beta-1} P(i,j)\{1-P_f(j)\}\left\{\sum_{m=1}^{n-1}[\{(1-\eta) + \eta(1-\xi)^m\right.$$
$$+ \sum_{\kappa=1}^{m}\eta\xi(1-\xi)^{(\kappa-1)}(1-\varsigma)\}\mathscr{A}_1(i,j,m)\left\{\sum_{\nu=1}^{\vartheta_m(j)}\right.$$
$$c_m(j)\rho_m(\nu,j)(V_1V_2)^{[T_L(i) + mT_i + \mathcal{T}_1(i,j,m) + Y_m(\nu,j)]}$$
$$\left. + c_l(i)T_m(j)(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + \mathcal{T}_1(i,j,m) + 0.5T_m(j)]}\right\}$$
$$+ \sum_{\kappa=1}^{m}\eta\xi(1-\xi)^{(m-1)}\varsigma\mathscr{A}_f(i,j,m)\mathscr{A}_4(i,j,m)\left\{\sum_{\nu=1}^{\vartheta_m(j)}\right.$$
$$c_m(j)\rho_m(\nu,j)(V_1V_2)^{[T_L(i) + mT_i + \mathcal{T}_4(i,j,m) + Y_m(\nu,j)]}$$

$$+ C_1(i)T_m(j)(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + \mathcal{T}_4(i,j,m) + 0.5T_m(j)]}\Big\} \tag{15}$$

$$+ \sum_{l=1}^{m}(\{(1-\eta) + \eta(1-\xi)^{(m-l)}\}A_1(i,j,m,l)\Big\{\sum_{\nu=l}^{\vartheta_m(j)} c_m$$

$$(j)\rho_m(\nu,j)(V_1V_2)^{[T_L(i) + mT_i + T_1(i,j,m,l) + \Upsilon_m(\nu,j)]} + c_1$$

$$(i)T_m(j)(V_1V_2)^{[T_L(i) + T(i,m) + mT_i + T_1(i,j,m,l) + 0.5T_m(j)]}\})]\Big\}$$

for $l=m\leq n\text{-}1$, $i\in\aleph$ and $i<j\in\Im$.

If the sub-system in state i makes a transition to any non-functional state $j\in\Im$, or to any functional state $j\in\Im$, after OCPM, before the warranty period $T_w(i)$, warranty recovery at the agreed rate is charged from the supplier/manufacturer [5]. Let the NPV of such a single step expected warranty charge recovered be $C_w(i)$; therefore, the net expected single step inspection cost, plus OCPM cost, plus on-failure minimal repair cost, minus warranty recovery $C_s(i)$, is:

$$C_s(i) = C_i(i) + C_o(i) + C_m(i) - C_w(i). \tag{16}$$

Therefore, the total expected single-step time taken by inspection, OCPM, and minimal repair, plus the time utilised for production, irrespective of the transition state and nature of the fault, given that the initial state is state i, is:

$$T_s(i,j) = \sum_{m=1}^{n-1}\Bigg[\Big\{\mathcal{P}_o(i,j,m) + \sum_{l=1}^{m}P_o(i,j,m,l)\Big\}T_o(i)$$

$$+ \Big\{\mathcal{P}_m(i,j,m) + \sum_{l=1}^{m}P_m(i,j,m,l)\Big\}T_m(i) \tag{17}$$

$$+ \Big\{\mathcal{A}_1(i,j,m) + \sum_{l=1}^{m}A_1(i,j,m,l)\Big\}mT_i + \mathcal{A}_1(i,j,m)T_1(i,j,m)$$

$$+ \sum_{l=1}^{m}A_1(i,j,m,l)T_1(i,j,m,l)\Bigg].$$

The NPV of the cumulative expected cost of inspection, expected cost of OCPM, expected cost of on-failure minimal repair, minus warranty recovery until the sub-system is in set $\aleph$, is:

$$C_c(i) = C_s(i) + \sum_{j=i+1}^{\beta-1} P(i,j)C_s(j)(V_1V_2)^{[T_s(i,j)]} + \sum_{j=i+1}^{\beta-2}\sum_{k=j+1}^{\beta-1} P(i,j)P(j,k)C_s$$

$$(k)(V_1V_2)^{[T_s(i,j) + T_s(j,k)]} + \sum_{j=i+1}^{\beta-3}\sum_{k=j+1}^{\beta-2}\sum_{l=k+1}^{\beta-1} P(i,j)P(j,k)P(k,l)C_s \tag{18}$$

$$(l)(V_1V_2)^{[T_s(i,j) + T_s(j,k) + T_s(k,l)]} + \ldots + P(i,j)P(j,k)\ldots P(\beta-2,\beta-1)C_s(\beta-1)$$

$$(V_1V_2)^{[T_s(i,j) + \ldots + T_s(\beta-2,\beta-1)]}$$

for $i\in\wp\subseteq\aleph<j<l\ldots<\beta\in\aleph$.

Here the present state and its status are self determined, while the inspection determines the sign of pending failure, and OCPM changes only the status of the next transition state, keeping its total probability of transition and transition time unchanged. Hence, the total time available for production $\mathcal{T}(i)$, before it makes the first transition to any state in the set $\aleph^c$, irrespective of the destination state, given that the process has started in state $i\epsilon\aleph$, is:

$$\mathcal{T}(i) = \sum_{j=1}^{\beta-1} \sum_{k=j+1}^{L} P_u(i,j)P(j,k) \sum_{m=1}^{n-1} \left[\mathcal{A}_1(j,k,m)\mathcal{T}_1(i,k,m) + \sum_{l=1}^{m} A_1(j,k,m,l)T_1(j,k,m,l) \right] \tag{19}$$

for $i\in\wp\subseteq\aleph$, $j\in\aleph$ and $k\in\aleph^c$.

Therefore, the total cumulative time consumed by inspection, OCPM and minimal repair plus the time utilised or available for production before the sub-system will make its first transition to any state in the set $\aleph^c$, given that the process has started in state i, is:

$$T_c(i) = \sum_{j=i}^{\beta-1} \sum_{k=j+1}^{L} P_u(i,j)P(j,k)T_s(j,k) \tag{20}$$

for $i\wp\subseteq\aleph$, $j\in\aleph$ and $k\in\aleph^c$.

Whenever the sub-system enters any state $k_*\epsilon\aleph^c$, its state is improved (either by repair or by replacement whichever is the most economical) to the better state $\delta(k_*)=i\epsilon\wp\subseteq\aleph$ as per policy. Thus, the time taken to implement the policy $T_L(i)$ and the NPV of the factory cost of the sub-system $C_p(i)$, are:

$$T_L(i) = \begin{cases} T_r(\delta(k_*)); \text{ for } \delta(k_*) = i \text{ and repair is economical} \\ T_p(i); \text{ for } \delta(k_*) = i \text{ and replacement is economical} \end{cases} \tag{21}$$

and

$$C_p(i) = \begin{cases} c_r(\delta(k_*)) \displaystyle\sum_{\nu=1}^{\nu_r(\delta(k_*))} \rho_r(\nu,\delta(K_*))(V_1V_2)^{[\Upsilon_r(\nu,\delta(k_*))]} + c_p(k_*) \\ \displaystyle\sum_{\nu=1}^{\vartheta_p(i)} \rho_p(\nu,k_*)(V_1V_2)^{[\Upsilon_p(\nu,k_*)]}; \text{ for economical repair} \\ c_p(i)\displaystyle\sum_{\nu=1}^{\vartheta_p(i)} \rho_p(\nu,i)(V_1V_2)^{[\Upsilon_p(\nu,i)]}; \text{ for economical replacement} \end{cases} \tag{22}$$

when $\delta(k_*)=i\in\wp\subseteq\aleph$ and $k_*\in\aleph^c$. $c_r(\delta(k_*))$, $c_p(i)$, $T_r(\delta(k_*))$ $T_p(i)$, $\vartheta_r(\delta(k_*))$, $\vartheta_p(i)$, $\Upsilon_r(\nu,\delta(k_*))$ and $\Upsilon_p(\nu,i)$ are repair and replacement costs, repair and replacement times, number of instalments and time for their payments for repair and replacement, respectively.

The probabilities of transition to the non-functional state j, $\mathcal{P}_m(i,j,m)$, and $P_m(i,j,m,l)$, during the mth inspection interval, $[T(i,m),T(i,m+1))$, irrespective of OCPM, provided that set 1 and set 2 conditions exist, respectively, are:

$$\mathcal{P}_m(i,j,m) = \begin{cases} 0 \text{ for } m > n-1,\ i\in\aleph \text{ and } i<j\in\Im \\ [\Big\{(1-\eta) + \eta\{1-\xi\}^m \\ + \displaystyle\sum_{\kappa=1}^{m} \eta\xi\{1-\xi\}^{(\kappa-1)}(1-\varsigma)\Big\}\mathcal{A}_1(i,j,m) + \sum_{\kappa=1}^{m} \eta\xi\{1-\xi\}^{(\kappa-1)}\varsigma \\ \mathcal{A}_1(i,j,m)\mathcal{A}_4(i,j,m)]\{1-P_f(j)\} \text{for } m\leq n-1,\ i\in\aleph \text{ and } i<j\in\Im \end{cases} \tag{23}$$

and

$$P_m(i,j,m,l) = \tag{24}$$

0 for $l > m$ and/or $m > n-1$, $iIN\aleph$ and $i<j\in\Im$

$[\{(1-\eta) + \eta\{1-\xi\}^{m-l}]A_1(i,j,m,l)\{1-P_f(j)\}$

for $l = m \leq n-1$, $i\in\aleph$ and $i<j\in\Im \Bigg[\Bigg\{(1-\eta) + \eta\{1-\xi\}^{(m-l)}$

$$+ \sum_{\kappa=l+1}^{m} \eta\xi\{1-\xi\}^{(\kappa-l-1)}(1-\varsigma)\}A_1(i,j,m,l)$$

$$+ \sum_{\kappa=l+1}^{m} \eta\xi\{1-\xi\}^{(\kappa-l-1)}\varsigma A_1(i,j,m,l)A_4(i,j,m,l)\Bigg]\{1-P_f(j)\}$$

for $l<m\leq n-1$, $i\in\aleph$ and $i<j\in\Im$

Since OCPM reduces the probability of transition to any non-functional state without changing the time of transition and the transition state, thus the net effective probabilities of transition of the sub-system in state i as a result of OCPM to any non-functional and functional states k_{nf} and $k_f\in\aleph_c$, are:

$$P_e(i,k_{nf}) = \sum_{j=i}^{\beta-1} P_u(i,j)P(i,k) \sum_{m=1}^{n-1}\left[\mathcal{P}_m(j,k,m) + \sum_{l=1}^{m} P_m(j,k,m,l)\right] \tag{25}$$

and

$$P_e(i,k_f) = \sum_{j=i}^{\beta-1} P_u(i,j)P(i,k) \sum_{m=1}^{n-1}\left[1-\left\{\mathcal{P}_m(j,k,m) + \sum_{l=1}^{m} P_m(j,k,m,l)\right\}\right] \tag{26}$$

for $i\in\wp\subseteq\aleph$, $j\in\aleph$ and k_{nf} and $k_f\in\aleph^c$. Where $P_u(i,j)$ represents the expected number of visits to state j, before it leaves the set $\aleph$ [7], given that the process has started in state i, the NPV of the expected cumulative capital cost of the sub-system at the time of the first transition to any state $k\epsilon\aleph_c$, irrespective of its status $C_M(i)$, is:

$$C_M(i) = C_p(i) - \sum_{k=\beta}^{L}\Bigg[P_e(i,k_f)c_p(k_f) \sum_{\nu=1}^{\vartheta_p(k_f)} \rho_p(\nu,k_f)(V_1V_2)^{[T_L(i)+T_c(i)+\Upsilon_p(\nu,k_f)]} \tag{27}$$

$$+ P_e(i,k_{nf})c_p(k_{nf}) \sum_{\nu=1}^{\vartheta_p(k_{nf})} \rho_p(\nu,k_{nf}) \; (V_1V_2)^{[T_L(i)+T_c(i)+\Upsilon_p(\nu,k_{nf})]}\Bigg]$$

for $i\in\wp\subseteq\aleph$, k_f and $k_{nf}\in\aleph^c$.

The expected availability of the sub-system $U(i)$ and the NPV of the expected cost rate per unit cycle time $C_r(i)$, until it makes transition to any state in the set $\aleph^c$, given that it is repaired or replaced in state $i\wp\subseteq\aleph$, at time $T_L(i)$, are:

$$U(i) = \left[\frac{\tau(i)}{T_L(i) + T_c(i)}\right] \tag{28}$$

and

$$C_r(i) = \left[\frac{C_c(i) + C_M(i)}{T_L(i) + T_c(i)}\right] \tag{29}$$

for $i \in \wp \subseteq \aleph$.

Since both the criteria for optimisation may be functions of the state in which the sub-system is repaired or replaced, the number of inspection intervals, and the control state β, the optimum purchase state i^*, the optimum control state β^* and the optimum number of inspection intervals (n^*-1) are those which result in the maximum expected availability of the sub-system $U_*(i)$, or the minimum cost rate/unit cycle time $C_r^*(i)$:

$$U^*(i) = \max_{i \in \Im} \max_{i < \beta \leq L} \max_{n \geq 2} U(i) \tag{30}$$

and

$$C_r^*(i) = \min_{i \in \Im} \min_{i < \beta \leq L} \min_{n \geq 2} C_r(i) \tag{31}$$

for $i \in \wp \subseteq \aleph$.

Let $\Pi(k_*)$ be the long run probability of transition to state $k_* \in \aleph^c$, the repair/replacement cost and its time are different for functional and non-functional states. Therefore, the optimum repair/replacement states $\delta^*(k_f)$ and $\delta^*(k_{nf})$ selected by the multi-state multi-level maintenance policy for functional state k_f and non-functional state k_{nf} may also be two different states. Thus, the expected long-run optimum availability of the sub-system $U(\bullet)$, and the long run NPV of the optimum expected cost rate per unit cycle time $C_r(\bullet)$ are:

$$U(\bullet) = \sum_{k \in \aleph^c} [\Pi(k_f)U^*(\delta(k_f)) + \Pi(k_{nf})U^*(\delta(k_{nf}))] \tag{32}$$

and

$$C_r(\bullet) = \sum_{k \in \aleph^c} [\Pi(k_f)C_r(\delta^*(k_f)) + \Pi(k_{nf})C_r(\delta^*(k_{nf}))] \tag{33}$$

for k_f and $k_{nf} \in \aleph^c$ and $\delta(k_f)$ and $\delta(k_{nf}) \in \wp \subseteq \aleph$.

The recommended cumulative capital recovery collected in the form of depreciation at any time T, which is measured with reference to the zero reference time, for initial state i, and the present state $k_* \in \Im$, is:

$$C_R(i) = c_p(i) \sum_{\nu=1}^{\vartheta_p(i)} \rho_p(\nu,i)(V_1V_2)^{[T_L(i) + T + \Upsilon_p(\nu,i)]} - c_p(k) \sum_{\nu=1}^{\vartheta_p(k_*)} \rho_p(\nu,k_*)(V_1V_2)^{[T_L(i) + T + \Upsilon_p(\nu,k_*)]} \tag{34}$$

for $i \in \wp \subseteq \aleph$, $k_* \in \Im$.

3. RESULTS AND CONCLUSIONS

A sub-system is assumed to be a multi-state multi-component sub-system (or a multi-state multi-dimensional single component sub-system), where the deterioration process is a semi-Markov process with a finite state space. The states are described in terms of a non-overlapping equivalent number of defects or an accumulated level of deterioration, and the nature of the process limits the occurrence of transitions to higher states. The present state of the sub-system is self-announcing, whereas inspection detects the sign of pending failure in terms of the transition state, its status and transition time. Emphasis is placed on a constant availability of the sub-system, in this respect inspection is scheduled in such a way that the risk of failure is a constant for each inspection interval.

In order to validate the mathematical model developed, and to provide a reasonably accurate decision tool which suggests optimum decisions within reasonable time, two alternative programs, Main-3 and Sim-3, were written in Turbo Pascal. These programs have a built-in provision for generating all of the alternative policies. Main-3 uses historical expected data, whereas Sim-3 uses self-generated simulated data based on historical data distributions. Each program when run calculates the optimal decision policy regarding: (i) the ideal state for purchase, (ii) the optimum control state, (iii) the optimum inspection schedule for each of the states $\in \aleph$ and (iv) the policy decision in case of transition to any higher functional state. Where the general policy decisions are: (a) to perform inspection as per determined schedule and keep the sub-system running until it makes a transition to any higher state or inspection detects pending failure (transition to a non-functional state) before the next scheduled inspection, (b) to perform on-condition preventive maintenance whenever inspection detects transition to a non-functional state before the next scheduled inspection, (c) to perform minimal repair if inspection fails to determine pending failure and transition occurs to a non-functional state less than the control state and (d) to perform repair or replacement to a better state whichever is economical, whenever the sub-system makes a transition to any functional or non-functional state higher than the control state as suggested by the policy.

Both programs were then run for a sub-system with a seven-member state space $\Im$ using a user-defined set of costs, probability of transition and time data. The results were then plotted in Figs 4–7. A comparison was then carried out and the following observations were made: (i) the optimum and the sub-optimum policies selected (Figs 4–5), the long run probability of transition to various states in the set $\aleph^c$ (Fig. 6) and the time available for a production cycle, i.e. the first passage time (Fig. 7), determined by both methods, were the same within reasonable limits of accuracy (Fig. 5) thereby validating the model

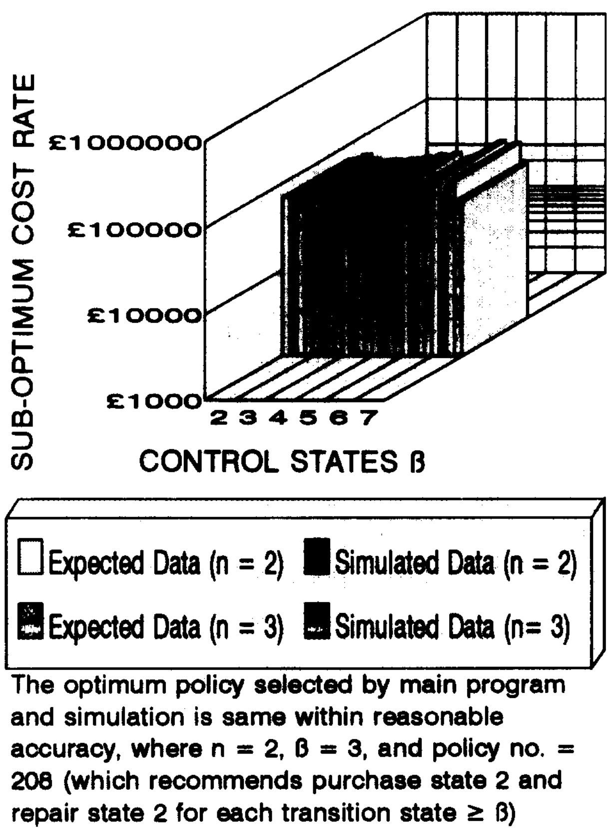

Fig. 4. Optimum maintenance cost rate.

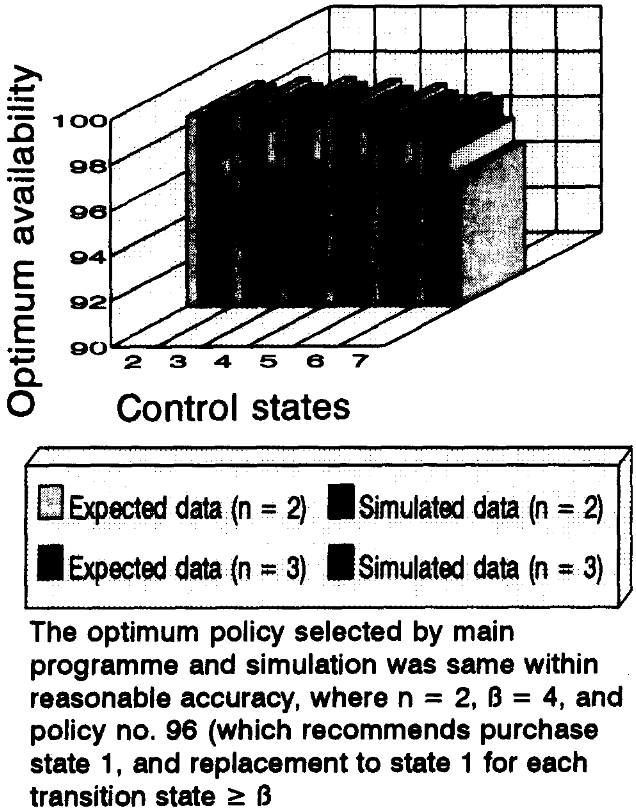

Fig. 5. Optimum system availability.

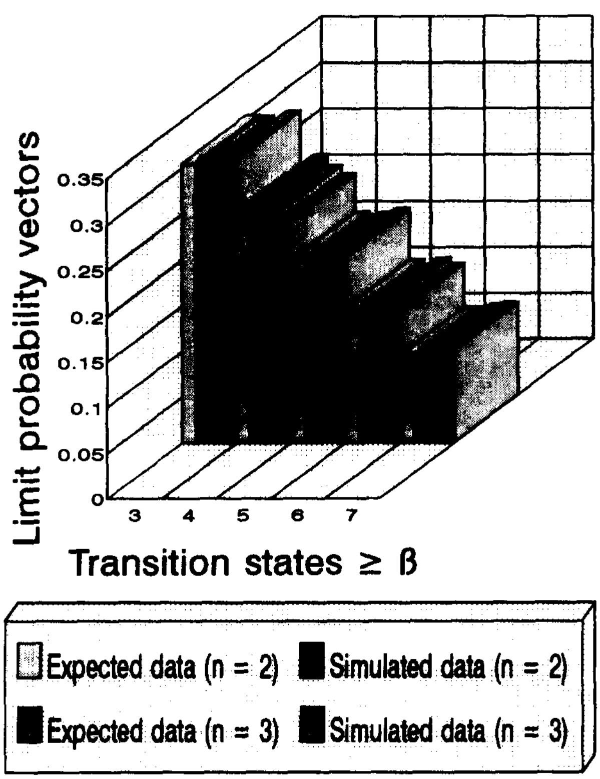

Fig. 6. The probability of transition to various states.

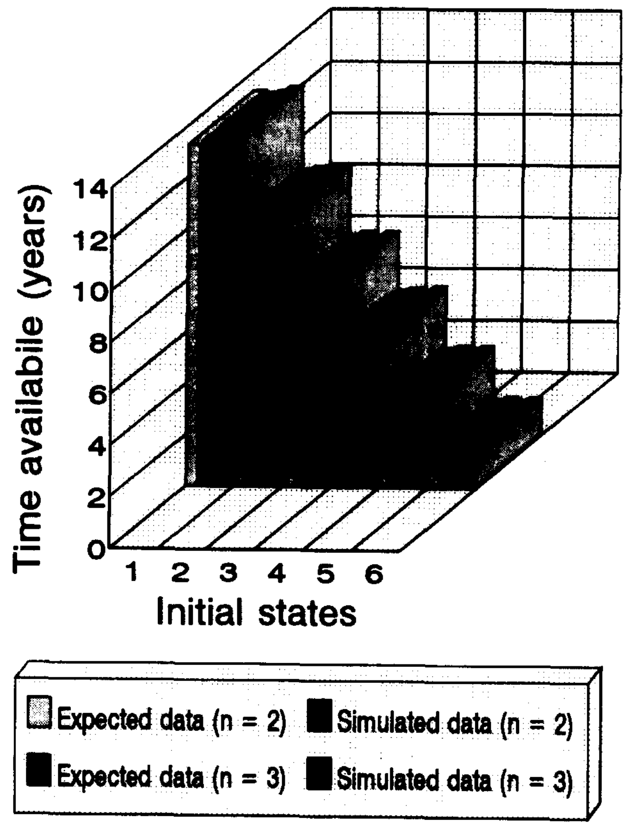

Fig. 7. Time available for a production cycle.

and the supporting software, (iii) the main program determined the optimum results within a few minutes whereas simulation runs took longer, and (iv) different policies were selected for different criteria because these decisions were data dependent (Figs 4 and 5). An industrial setup was modelled using the ProModel simulation package. To facilitate decision making, user-defined sub-routines were written in Turbo Pascal. The ProModel simulation was then run and identical results were obtained; this confirms the validity of the model and the supporting software.

Capital recovery for a production system is taken to be equal to the drop in value considering inflation or deflation, which enables mature as well as premature replacement by the same or its alternative without additional investment.

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